Wigner distribution function applied to twisted Gaussian light propagating in first-order optical systems

Martin J. Bastiaans

Technische Universiteit Eindhoven, Faculteit Elektrotechniek
Postbus 513, 5600 MB Eindhoven, Netherlands

Abstract

A measure for the twist of Gaussian light is expressed in terms of the second-order moments of the Wigner distribution function. The propagation law for these second-order moments between the input plane and the output plane of a first-order optical system is used to express the twist in one plane in terms of moments in the other plane. Although in general the twist in one plane is determined not only by the twist in the other plane, but also by other combinations of the moments, several special cases exist for which a direct relationship between the twists can be formulated. Three such cases, for which zero twist is preserved, are considered: (i) propagation between conjugate planes, (ii) adaptation of the signal to the system, and (iii) the case of symplectic Gaussian light.

OCIS codes: 070.2580, 080.2730, 030.5630

1 Introduction

In recent years there has been some interest in the twist of Gaussian light [1, 2, 3, 4, 5, 6, 7]. In this paper we consider the propagation of this twist in first-order optical systems and we use the Wigner distribution function as a mathematical tool to do so. Preliminary versions [8, 9] of this paper were presented at ICOL '98, the International Conference on Optics and Optoelectronics, Dehradun, India, 1998, and at ICO-XVIII, the 18th Congress of the International Commission for Optics, San Francisco, CA, 1999.

In Section 2 we first represent Gaussian light by means of its cross-spectral density and by means of its Wigner distribution function. Moreover, we introduce the moments of the Wigner distribution function and we define a measure for the twist, based on these moments. In Section 3 we introduce a first-order optical system, and describe its input-output relationship in terms of a ray transformation matrix as well as in terms of its coherent point-spread function. The propagation of the moments of the Wigner distribution function through first-order optical systems is presented in Section 4, which section leads to a general relationship between the twist in the output plane and the moments in the input plane, and vice versa. Although the general relationships do not show an easy interpretation, they will form a basis to consider some special cases in Section 5.

Before we start, we make some remarks about notation. We throughout consider an optical signal in a plane z = constant; the signal thus depends on the transverse coordinates x and y, only, which coordinates are combined into a 2-dimensional column vector $\boldsymbol{r}$. The spatial-frequency variable is denoted by the 2-dimensional column vector $\boldsymbol{q}$, whereas the temporal-frequency dependence of the optical signal will not be taken into account, since it is not relevant in the context of this paper. While vectors are denoted by bold-face, lower-case characters, matrices are throughout be denoted by bold-face, upper-case characters. Finally, the superscript ^{t} is used to denote transposition of vectors and matrices.

2 Representation of Gaussian light

The cross-spectral density [10, 11, 12] of the most general partially coherent Gaussian light can be written in the form [13]

$$\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\pi}\sqrt{\det \boldsymbol{G}_1}\exp\left(-\tfrac{1}{4}\begin{bmatrix} \boldsymbol{r}_1+\boldsymbol{r}_2 \\ \boldsymbol{r}_1-\boldsymbol{r}_2 \end{bmatrix}^t \begin{bmatrix} \boldsymbol{G}_1 & -i\boldsymbol{H} \\ -i\boldsymbol{H}^t & \boldsymbol{G}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{r}_1+\boldsymbol{r}_2 \\ \boldsymbol{r}_1-\boldsymbol{r}_2 \end{bmatrix}\right), \tag{1}$$

where we have chosen a representation that enables us to determine the Wigner distribution function of such light in an easy way. The exponent shows a quadratic form in which a 4-dimensional column vector $[(\boldsymbol{r}_1+\boldsymbol{r}_2)^t,(\boldsymbol{r}_1-\boldsymbol{r}_2)^t]^t$ arises, together with a 4×4 symmetric matrix. This matrix consists of four real $n\times n$ submatrices $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, $\boldsymbol{H}$, and $\boldsymbol{H}^t$, where, moreover, the matrices $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ (as well as the matrix $\boldsymbol{G}_2-\boldsymbol{G}_1$) are positive definite symmetric. The special form of the matrix is a direct consequence of the fact that the cross-spectral density is a nonnegative definite Hermitian function [11, 12].

In a more common way, the cross-spectral density of Gaussian light can be expressed in the form

$$\begin{aligned}\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\pi}\sqrt{\det \boldsymbol{G}_1} & \exp\{-\tfrac{1}{4}(\boldsymbol{r}_1-\boldsymbol{r}_2)^t(\boldsymbol{G}_2-\boldsymbol{G}_1)(\boldsymbol{r}_1-\boldsymbol{r}_2)\} \\ \times & \exp\{-\tfrac{1}{2}\boldsymbol{r}_1^t[\boldsymbol{G}_1-i\tfrac{1}{2}(\boldsymbol{H}+\boldsymbol{H}^t)]\boldsymbol{r}_1\} \\ \times & \exp\{-\tfrac{1}{2}\boldsymbol{r}_2^t[\boldsymbol{G}_1+i\tfrac{1}{2}(\boldsymbol{H}+\boldsymbol{H}^t)]\boldsymbol{r}_2\} \\ \times & \exp\{-\tfrac{1}{2}\boldsymbol{r}_1^t i(\boldsymbol{H}-\boldsymbol{H}^t)\boldsymbol{r}_2\}.\end{aligned} \tag{2}$$

Note that the asymmetry of the matrix $\boldsymbol{H}$ is a measure for the twist [1, 2, 3, 4, 5, 6, 7] of Gaussian light, and that general Gaussian light reduces to zero-twist Gaussian Schell-model light [14, 15], if the matrix $\boldsymbol{H}$ is symmetric, $\boldsymbol{H}-\boldsymbol{H}^t=\boldsymbol{0}$. In that case the light can be considered as originating from spatially stationary light with a Gaussian cross-spectral density $\exp\{-\frac{1}{4}(\boldsymbol{r}_1-\boldsymbol{r}_2)^t(\boldsymbol{G}_2-\boldsymbol{G}_1)(\boldsymbol{r}_1-\boldsymbol{r}_2)\}$, modulated by a Gaussian modulator with modulation function $\exp\{-\frac{1}{2}\boldsymbol{r}^t(\boldsymbol{G}_1-i\boldsymbol{H})\boldsymbol{r}\}$. Note that the matrix $\boldsymbol{G}_1$ represents the Gaussian-shaped intensity profile of the light, the matrix $\boldsymbol{G}_2-\boldsymbol{G}_1$ the Gaussian-shaped coherence properties of the light (with the light completely coherent if $\boldsymbol{G}_2-\boldsymbol{G}_1=\boldsymbol{0}$), and the matrix $\boldsymbol{H}$ the curvature of the light.

In terms of the Wigner distribution function [16], which is defined as the spatial Fourier transform of the cross-spectral density $\Gamma(\boldsymbol{r}+\frac{1}{2}\boldsymbol{r}',\boldsymbol{r}-\frac{1}{2}\boldsymbol{r}')$ with respect to the difference coordinate $\boldsymbol{r}'$ [13, 17, 18],

$$F(\boldsymbol{r},\boldsymbol{q}) = \int_{-\infty}^{\infty}\Gamma(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{r}',\boldsymbol{r}-\tfrac{1}{2}\boldsymbol{r}')\exp\{-i\boldsymbol{q}^t\boldsymbol{r}'\}d\boldsymbol{r}', \tag{3}$$

Gaussian light can be represented as [13, 19, 20]

$$F(\boldsymbol{r},\boldsymbol{q}) = 4\sqrt{\frac{\det \boldsymbol{G}_1}{\det \boldsymbol{G}_2}} \exp\left(- \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{q} \end{bmatrix}^t \begin{bmatrix} \boldsymbol{G}_1 + \boldsymbol{H}\boldsymbol{G}_2^{-1}\boldsymbol{H}^t & -\boldsymbol{H}\boldsymbol{G}_2^{-1} \\ -\boldsymbol{G}_2^{-1}\boldsymbol{H}^t & \boldsymbol{G}_2^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{q} \end{bmatrix} \right). \tag{4}$$

Whereas the (Hermitian) cross-spectral density $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2)$ describes an optical signal in the space domain, the (real) Wigner distribution function $F(\boldsymbol{r},\boldsymbol{q})$ describes the signal in a so-called phase space, i.e. a combined space-frequency domain. The Wigner distribution function thus resembles the ray concept in geometrical optics, where the position and direction of an optical ray are given simultaneously, too. In a way, $F(\boldsymbol{r},\boldsymbol{q})$ is the amplitude of a ray that passes through the position $\boldsymbol{r}$ and has a direction (or spatial frequency) $\boldsymbol{q}$.

The $2n \times 2n$ real symmetric matrix $\boldsymbol{M}$ of normalized second-order moments of the Wigner distribution function is defined by [13, 18, 20]

$$\boldsymbol{M} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\boldsymbol{r},\boldsymbol{q}) d\boldsymbol{r} d\frac{\boldsymbol{q}}{2\pi} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \begin{bmatrix} \boldsymbol{r}\boldsymbol{r}^t & \boldsymbol{r}\boldsymbol{q}^t \\ \boldsymbol{q}\boldsymbol{r}^t & \boldsymbol{q}\boldsymbol{q}^t \end{bmatrix} F(\boldsymbol{r},\boldsymbol{q}) d\boldsymbol{r} d\frac{\boldsymbol{q}}{2\pi}. \tag{5}$$

It yields not only such quantities as the commonly known effective spatial width of a light beam, corresponding to the second-order moments of the spatial variable $\boldsymbol{r}$, and the effective directional width of the beam, corresponding to the second-order moments of the directional variable $\boldsymbol{q}$, but also mixed quantities, which follow from the products $\boldsymbol{r}\boldsymbol{q}^t$ and $\boldsymbol{q}\boldsymbol{r}^t$. It can be shown [13] that the symmetric moment matrix $\boldsymbol{M}$ is positive definite; an easy proof of this property is presented in Appendix A. Note, however, that although positivity of the symmetric moment matrix $\boldsymbol{M}$ is a necessary condition for and a consequence from the nonnegativity of the Hermitian cross-spectral density $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2)$, it is not a sufficient condition for this nonnegativity. Other and more stronger conditions have been presented in [21] and [22].

In the case of Gaussian light the moment matrix $\boldsymbol{M}$ takes the form

$$\begin{aligned} \boldsymbol{M} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{P} \\ \boldsymbol{P}^t & \boldsymbol{Q} \end{bmatrix} &= \tfrac{1}{2} \begin{bmatrix} \boldsymbol{G}_1 + \boldsymbol{H}\boldsymbol{G}_2^{-1}\boldsymbol{H}^t & -\boldsymbol{H}\boldsymbol{G}_2^{-1} \\ -\boldsymbol{G}_2^{-1}\boldsymbol{H}^t & \boldsymbol{G}_2^{-1} \end{bmatrix}^{-1} \\ &= \tfrac{1}{2} \begin{bmatrix} \boldsymbol{G}_1^{-1} & \boldsymbol{G}_1^{-1}\boldsymbol{H} \\ \boldsymbol{H}^t\boldsymbol{G}_1^{-1} & \boldsymbol{G}_2 + \boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{H} \end{bmatrix} \end{aligned} \tag{6}$$

and hence

$$\boldsymbol{R} = \tfrac{1}{2}\boldsymbol{G}_1^{-1}, \quad \boldsymbol{P} = \tfrac{1}{2}\boldsymbol{G}_1^{-1}\boldsymbol{H}, \quad \boldsymbol{Q} = \tfrac{1}{2}(\boldsymbol{G}_2 + \boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{H}). \tag{7}$$

Note that the positive definiteness of the matrix $\boldsymbol{M}$ leads to the positivity of the quadratic form

$$2 \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{r} \end{bmatrix}^t \boldsymbol{M} \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{r} \end{bmatrix} = \boldsymbol{r}^t\boldsymbol{G}_2\boldsymbol{r} + (\boldsymbol{q} + \boldsymbol{H}\boldsymbol{r})^t\boldsymbol{G}_1^{-1}(\boldsymbol{q} + \boldsymbol{H}\boldsymbol{r}) \tag{8}$$

for any vectors $\boldsymbol{r}$ and $\boldsymbol{q}$, which immediately leads to the positive definiteness of the matrices $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$, as already mentioned.

From Eqs. (7) we conclude that the matrices $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{H}$ follow directly from the moment matrices $\boldsymbol{R}$, $\boldsymbol{Q}$, and $\boldsymbol{P}$:

$$\boldsymbol{G}_1 = \tfrac{1}{2}\boldsymbol{R}^{-1}, \quad \boldsymbol{G}_2 = 2(\boldsymbol{Q} - \boldsymbol{P}^t\boldsymbol{R}^{-1}\boldsymbol{P}), \quad \boldsymbol{H} = \boldsymbol{R}^{-1}\boldsymbol{P}. \tag{9}$$

As a consequence we can express the twistedness of the Gaussian light directly in terms of the moment matrices $\boldsymbol{R}$ and $\boldsymbol{P}$:

$$\boldsymbol{PR} - \boldsymbol{RP}^t = \boldsymbol{R}(\boldsymbol{H} - \boldsymbol{H}^t)\boldsymbol{R} = \tfrac{1}{4}\boldsymbol{G}_1^{-1}(\boldsymbol{H} - \boldsymbol{H}^t)\boldsymbol{G}_1^{-1} = \tfrac{1}{4}(\boldsymbol{H} - \boldsymbol{H}^t)\det \boldsymbol{G}_1^{-1}. \tag{10}$$

3 Description of a first-order optical system

In this section we consider the propagation of Gaussian light through a first-order optical system [23], described by the input-output relationship [13, 18]

$$F_o(\boldsymbol{r},\boldsymbol{q}) = F_i(\boldsymbol{Ar}+\boldsymbol{Bq},\boldsymbol{Cr}+\boldsymbol{Dq}), \tag{11}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are real 2×2 matrices. For such a first-order optical system a single input ray, entering the system at the position $\boldsymbol{r}_i$ with direction $\boldsymbol{q}_i$, yields a single output ray, leaving the system at the position $\boldsymbol{r}_o$ with direction $\boldsymbol{q}_o$. The input and output positions and directions are related by the matrix relationship [23]

$$\begin{bmatrix} \boldsymbol{r}_i \\ \boldsymbol{q}_i \end{bmatrix} = \boldsymbol{T}\begin{bmatrix} \boldsymbol{r}_o \\ \boldsymbol{q}_o \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}\begin{bmatrix} \boldsymbol{r}_o \\ \boldsymbol{q}_o \end{bmatrix}. \tag{12}$$

The latter relation is a well-known geometric-optical matrix description of a first-order optical system; the 4×4 $\boldsymbol{ABCD}$ matrix in this relationship is known as the ray transformation matrix $\boldsymbol{T}$ [24]. The ray transformation matrix is symplectic [13, 18, 23, 24], which implies the relations

$$\boldsymbol{AB}^t = \boldsymbol{BA}^t, \quad \boldsymbol{C}^t\boldsymbol{A} = \boldsymbol{A}^t\boldsymbol{C}, \quad \boldsymbol{AD}^t - \boldsymbol{BC}^t = \mathbf{I}, \tag{13}$$

or, equivalently,

$$\boldsymbol{B}^t\boldsymbol{D} = \boldsymbol{D}^t\boldsymbol{B}, \quad \boldsymbol{DC}^t = \boldsymbol{CD}^t, \quad \boldsymbol{A}^t\boldsymbol{D} - \boldsymbol{C}^t\boldsymbol{B} = \mathbf{I}. \tag{14}$$

In a different way, a first-order optical system, like any linear optical system, can be described by means of the input-output relationship

$$\Gamma_o(\boldsymbol{r}_{1o},\boldsymbol{r}_{2o}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\boldsymbol{r}_{1o},\boldsymbol{r}_{1i})\Gamma_i(\boldsymbol{r}_{1i},\boldsymbol{r}_{2i})h^*(\boldsymbol{r}_{2o},\boldsymbol{r}_{2i})d\boldsymbol{r}_{1i}d\boldsymbol{r}_{2i}, \tag{15}$$

where we have introduced the coherent point-spread function $h(\boldsymbol{r}_o,\boldsymbol{r}_i)$. In the case of a first-order optical system, this coherent point-spread function is proportional to [25]

$$\exp\left(i\tfrac{1}{2}\begin{bmatrix} \boldsymbol{r}_o \\ \boldsymbol{r}_i \end{bmatrix}^t \begin{bmatrix} \boldsymbol{L}_{oo} & \boldsymbol{L}_{oi} \\ \boldsymbol{L}_{io} & \boldsymbol{L}_{ii} \end{bmatrix}\begin{bmatrix} \boldsymbol{r}_o \\ \boldsymbol{r}_i \end{bmatrix}\right), \tag{16}$$

where the four real 2×2 matrices $\boldsymbol{L}_{oo}$, $\boldsymbol{L}_{oi}$, $\boldsymbol{L}_{io}$, and $\boldsymbol{L}_{ii}$ constitute a matrix $\boldsymbol{L}$ which, without loss of generality, can be assumed to be symmetric: an antisymmetric part of $\boldsymbol{L}$ would not contribute to the quadratic form in expression (16). The relationship between the input and output positions and directions of a single ray can now be expressed in terms of the matrix $\boldsymbol{L}$ [25]:

$$\begin{bmatrix} \boldsymbol{q}_o \\ -\boldsymbol{q}_i \end{bmatrix} = \boldsymbol{L}\begin{bmatrix} \boldsymbol{r}_o \\ \boldsymbol{r}_i \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}_{oo} & \boldsymbol{L}_{oi} \\ \boldsymbol{L}_{io} & \boldsymbol{L}_{ii} \end{bmatrix}\begin{bmatrix} \boldsymbol{r}_o \\ \boldsymbol{r}_i \end{bmatrix}. \tag{17}$$

The equivalence of the two input-output relationships (12) and (17) allows to express the elements of the symmetric matrix $\boldsymbol{L}$ in terms of the elements of the symplectic ray transformation matrix $\boldsymbol{T}$, [25]

$$\boldsymbol{L}_{oo} = \boldsymbol{L}_{oo}^t = -\boldsymbol{B}^{-1}\boldsymbol{A}, \quad \boldsymbol{L}_{oi} = \boldsymbol{L}_{io}^t = \boldsymbol{B}^{-1}, \quad \boldsymbol{L}_{ii} = \boldsymbol{L}_{ii}^t = -\boldsymbol{D}\boldsymbol{B}^{-1}, \tag{18}$$

where we have assumed that the matrix $\boldsymbol{B}$ is non-singular. A singular matrix $\boldsymbol{B}$ would correspond to a degenerate form of the quadratic form in expression (16) and would lead to Dirac functions in the coherent point-spread functions: $\delta(x) = \lim_{\alpha\to\infty}\sqrt{\alpha/2\pi i}\,\exp(i\frac{1}{2}\alpha x^2)$.

4 Propagation of the moments

The propagation of the moment matrix $\boldsymbol{M}$ through a first-order optical system with ray transformation matrix $\boldsymbol{T}$ can be described by the relationship [13, 18, 20, 26, 27]

$$\boldsymbol{M}_i = \boldsymbol{T}\boldsymbol{M}_o\boldsymbol{T}^t. \tag{19}$$

This relationship can readily be derived by combining the input-output relationship (12) of the first-order optical system with the definition (5) of the moment matrices $\boldsymbol{M}_i$ and $\boldsymbol{M}_o$ of the input and the output signal, respectively.

In terms of the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{P}$, $\boldsymbol{Q}$, and $\boldsymbol{R}$, Eq. (19) takes the form

$$\begin{bmatrix} \boldsymbol{R}_i & \boldsymbol{P}_i \\ \boldsymbol{P}_i^t & \boldsymbol{Q}_i \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_o & \boldsymbol{P}_o \\ \boldsymbol{P}_o^t & \boldsymbol{Q}_o \end{bmatrix} \begin{bmatrix} \boldsymbol{A}^t & \boldsymbol{C}^t \\ \boldsymbol{B}^t & \boldsymbol{D}^t \end{bmatrix} \tag{20}$$

and hence

$$\begin{array}{rcl} \boldsymbol{R}_i &=& \boldsymbol{A}\boldsymbol{R}_o\boldsymbol{A}^t + \boldsymbol{A}\boldsymbol{P}_o\boldsymbol{B}^t + \boldsymbol{B}\boldsymbol{P}_o^t\boldsymbol{A}^t + \boldsymbol{B}\boldsymbol{Q}_o\boldsymbol{B}^t \\ \boldsymbol{P}_i &=& \boldsymbol{A}\boldsymbol{R}_o\boldsymbol{C}^t + \boldsymbol{A}\boldsymbol{P}_o\boldsymbol{D}^t + \boldsymbol{B}\boldsymbol{P}_o^t\boldsymbol{C}^t + \boldsymbol{B}\boldsymbol{Q}_o\boldsymbol{D}^t \\ \boldsymbol{Q}_i &=& \boldsymbol{C}\boldsymbol{R}_o\boldsymbol{C}^t + \boldsymbol{C}\boldsymbol{P}_o\boldsymbol{D}^t + \boldsymbol{D}\boldsymbol{P}_o^t\boldsymbol{C}^t + \boldsymbol{D}\boldsymbol{Q}_o\boldsymbol{D}^t. \end{array} \tag{21}$$

Using Eqs. (7), (9), and (21), we can formulate relationships between the matrices $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{H}$ in the input and the output plane. In particular we can express the input twist in terms of the output matrices $\boldsymbol{G}_{1o}$, $\boldsymbol{G}_{2o}$, and $\boldsymbol{H}_o$, and the output twist in terms of the input matrices $\boldsymbol{G}_{1i}$, $\boldsymbol{G}_{2i}$, and $\boldsymbol{H}_i$ (see Appendix B):

$$\begin{array}{rcl} \boldsymbol{G}_{1i}^{-1}(\boldsymbol{H}_i - \boldsymbol{H}_i^t)\boldsymbol{G}_{1i}^{-1} &=& (\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}_o^t)\boldsymbol{G}_{1o}^{-1}(\boldsymbol{H}_o - \boldsymbol{H}_o^t)\boldsymbol{G}_{1o}^{-1}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}_o^t)^t \\ &-& (\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}_o^t)\boldsymbol{G}_{1o}^{-1}\boldsymbol{G}_{2o}\boldsymbol{B}^t + \boldsymbol{B}\boldsymbol{G}_{2o}\boldsymbol{G}_{1o}^{-1}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}_o^t)^t. \end{array} \tag{22}$$

$$\begin{array}{rcl} \boldsymbol{G}_{1o}^{-1}(\boldsymbol{H}_o - \boldsymbol{H}_o^t)\boldsymbol{G}_{1o}^{-1} &=& (\boldsymbol{D}-\boldsymbol{H}_i\boldsymbol{B})^t\boldsymbol{G}_{1i}^{-1}(\boldsymbol{H}_i - \boldsymbol{H}_i^t)\boldsymbol{G}_{1i}^{-1}(\boldsymbol{D}-\boldsymbol{H}_i\boldsymbol{B}) \\ &+& (\boldsymbol{D}-\boldsymbol{H}_i\boldsymbol{B})^t\boldsymbol{G}_{1i}^{-1}\boldsymbol{G}_{2i}\boldsymbol{B} - \boldsymbol{B}^t\boldsymbol{G}_{2i}\boldsymbol{G}_{1i}^{-1}(\boldsymbol{D}-\boldsymbol{H}_i\boldsymbol{B}). \end{array} \tag{23}$$

Note that zero twist in the output plane, $\boldsymbol{H}_o = \boldsymbol{H}_o^t$, corresponds to zero twist in the input plane, $\boldsymbol{H}_i = \boldsymbol{H}_i^t$, if

$$\begin{array}{rcl} (\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}_o^t)\boldsymbol{G}_{1o}^{-1}\boldsymbol{G}_{2o}\boldsymbol{B}^t &=& \boldsymbol{B}\boldsymbol{G}_{2o}\boldsymbol{G}_{1o}^{-1}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}_o^t)^t \\ (\boldsymbol{D}-\boldsymbol{H}_i\boldsymbol{B})^t\boldsymbol{G}_{1i}^{-1}\boldsymbol{G}_{2i}\boldsymbol{B} &=& \boldsymbol{B}^t\boldsymbol{G}_{2i}\boldsymbol{G}_{1i}^{-1}(\boldsymbol{D}-\boldsymbol{H}_i\boldsymbol{B}); \end{array} \tag{24}$$

in general, however, zero twist is not preserved. Three special cases for which the conditions (24) are met, are obvious.

5 Special cases

The relationships (22) and (23) show that in general the twist in one plane is determined not only by the twist in the other plane, i.e. by the combination $\boldsymbol{G}_1^{-1}(\boldsymbol{H}-\boldsymbol{H}^t)\boldsymbol{G}_1^{-1}$, but by other combinations of the matrices $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{H}$, as well. However, there are some special cases in which a more direct relationship between the twists in the input and the output plane can be formulated.

5.1 Propagation between conjugate planes: $\boldsymbol{B}=\mathbf{0}$

For the special first-order optical system for which $\boldsymbol{B}=\mathbf{0}$ [and hence also, due to the condition of symplecticity, $\boldsymbol{A}\boldsymbol{D}^t=\mathbf{I}$ and $\boldsymbol{C}^t\boldsymbol{A}=\boldsymbol{A}^t\boldsymbol{C}$, see Eq. (13)], i.e. when we consider propagation between two conjugate planes [28], we have simple relationships between the input and output matrices $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{H}$:

$$\boldsymbol{G}_{1o}=\boldsymbol{A}^t\boldsymbol{G}_{1i}\boldsymbol{A},\quad \boldsymbol{G}_{2o}=\boldsymbol{A}^t\boldsymbol{G}_{2i}\boldsymbol{A},\quad \boldsymbol{H}_o=\boldsymbol{A}^t\boldsymbol{H}_i\boldsymbol{A}-\boldsymbol{C}^t\boldsymbol{A}. \tag{25}$$

These relationships follow directly from combining the matrix relations (7) and (9) with the propagation relations (21) for $\boldsymbol{B}=\mathbf{0}$, while using the symplecticity conditions (13) and (14). Note in particular that $\boldsymbol{H}_o-\boldsymbol{H}_o^t=\boldsymbol{A}^t(\boldsymbol{H}_i-\boldsymbol{H}_i^t)\boldsymbol{A}=(\boldsymbol{H}_i-\boldsymbol{H}_i^t)\det\boldsymbol{A}$, and that the optical system does not change the twistedness of the signal: zero twist is preserved.

5.2 Adapted signals: $\boldsymbol{D}-\boldsymbol{H}_i\boldsymbol{B}=\mathbf{0}$ and $\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}_o^t=\mathbf{0}$

If $\boldsymbol{B}\neq\mathbf{0}$, simple relationships of the form (25) no longer exist. However, for any first-order optical system (with the mild condition $\det\boldsymbol{B}\neq 0$) there exists a certain signal for which, again, special relationships can be formulated.

In the special case that the output signal (with matrix $\boldsymbol{H}_o$) is adapted to the system according to

$$\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}_o^t=\mathbf{0}, \tag{26}$$

Eq. (22) immediately shows that the input twist is zero: $\boldsymbol{H}_i-\boldsymbol{H}_i^t=\mathbf{0}$. We remark that, due to the symplecticity condition $\boldsymbol{A}\boldsymbol{B}^t=\boldsymbol{B}\boldsymbol{A}^t$ [see Eq. (13)], the adaptation condition (26) implies $\boldsymbol{B}(\boldsymbol{H}_o-\boldsymbol{H}_o^t)\boldsymbol{B}^t=\mathbf{0}$ and hence $(\boldsymbol{H}_o-\boldsymbol{H}_o^t)\det\boldsymbol{B}=\mathbf{0}$. Similar results follow from Eq. (23) and the symplecticity condition $\boldsymbol{B}^t\boldsymbol{D}=\boldsymbol{D}^t\boldsymbol{B}$ [see Eq. (14)], for the case that the input signal (with matrix $\boldsymbol{H}_i$) is adapted to the system according to

$$\boldsymbol{D}-\boldsymbol{H}_i\boldsymbol{B}=\mathbf{0}, \tag{27}$$

in which case we have $\boldsymbol{H}_o-\boldsymbol{H}_o^t=\mathbf{0}$ and $(\boldsymbol{H}_i-\boldsymbol{H}_i^t)\det\boldsymbol{B}=\mathbf{0}$.

Let us assume that the output signal is adapted. If we substitute from Eq. (26) and from the relation $\boldsymbol{P}_o=\boldsymbol{R}_o\boldsymbol{H}_o$ [cf. Eq. (9)] into Eqs. (21), we get

$$\begin{aligned}\boldsymbol{R}_i &= \boldsymbol{B}\boldsymbol{Q}_o\boldsymbol{B}^t-\boldsymbol{A}\boldsymbol{R}_o\boldsymbol{A}^t\\ \boldsymbol{P}_i &= \boldsymbol{B}\boldsymbol{Q}_o\boldsymbol{D}^t+\boldsymbol{A}\boldsymbol{R}_o\boldsymbol{H}_o\boldsymbol{D}^t.\end{aligned} \tag{28}$$

Using the adaptation condition (26) and the symplecticity condition $\boldsymbol{B}^t\boldsymbol{D} = \boldsymbol{D}^t\boldsymbol{B}$ again, we are immediately led to the relationship

$$\boldsymbol{P}_i\boldsymbol{B} = \boldsymbol{R}_i\boldsymbol{D}. \tag{29}$$

With $\boldsymbol{P}_i = \boldsymbol{R}_i\boldsymbol{H}_i$ [cf. Eq. (9)], Eq. (29) takes the form $\boldsymbol{R}_i(\boldsymbol{D} - \boldsymbol{H}_i\boldsymbol{B}) = \mathbf{0}$, and we conclude that the input signal is adapted as well, see Eq. (27).

We are thus led to the important conclusion that for every first-order optical system there exists a signal which is adapted in both the input and the output plane, and has zero twist in these planes if $\det \boldsymbol{B} \neq 0$. In general, however, a zero-twist signal in one plane does not correspond to a zero-twist signal in the other plane. It is not too difficult to show that in the case of adaptation the relationships between the input and output matrices $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ again take simple forms:

$$\boldsymbol{G}_{2o}^{-1} = \boldsymbol{B}^t\boldsymbol{G}_{1i}\boldsymbol{B}, \quad \boldsymbol{G}_{1o}^{-1} = \boldsymbol{B}^t\boldsymbol{G}_{2i}\boldsymbol{B}. \tag{30}$$

The process of adaptation becomes more clear if we consider the adaptation conditions (26) and (27) in relation to the input-output description of the first-order optical system by means of its coherent point-spread function (16). We note that in the case of adaptation the input signal is indeed adapted to the system in that the curvature $\boldsymbol{H}_i$ of the Gaussian input signal cancels one of the quadratic terms, $\boldsymbol{L}_{ii} = -\boldsymbol{D}\boldsymbol{B}^{-1}$, in the exponent of the coherent point-spread function, and that the output signal is adapted in that the curvature $\boldsymbol{H}_o$ of the Gaussian output signal equals the other quadratic term, $\boldsymbol{L}_{oo} = -\boldsymbol{B}^{-1}\boldsymbol{A}$, in this exponent. The output cross-spectral density thus becomes proportional to

$$\begin{aligned}&\exp\left\{i\tfrac{1}{2}\left(\boldsymbol{r}_{1o}^t\boldsymbol{H}_o\boldsymbol{r}_{1o} - \boldsymbol{r}_{2o}^t\boldsymbol{H}_o\boldsymbol{r}_{2o}\right)\right\}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{i\left(\boldsymbol{r}_{1o}^t\boldsymbol{L}_{oi}\boldsymbol{r}_{1i} - \boldsymbol{r}_{2o}^t\boldsymbol{L}_{oi}\boldsymbol{r}_{2i}\right)\right\}\\ &\times\ \exp\left\{-\tfrac{1}{2}\boldsymbol{r}_{1i}^t\boldsymbol{G}_{1i}\boldsymbol{r}_{1i} - \tfrac{1}{4}(\boldsymbol{r}_{1i} - \boldsymbol{r}_{2i})^t(\boldsymbol{G}_{2i} - \boldsymbol{G}_{1i})(\boldsymbol{r}_{1i} - \boldsymbol{r}_{2i}) - \tfrac{1}{2}\boldsymbol{r}_{2i}^t\boldsymbol{G}_{1i}\boldsymbol{r}_{2i}\right\}d\boldsymbol{r}_{1i}d\boldsymbol{r}_{2i}\end{aligned}$$

and we note that the integral represents a Fourier transformation, involving only the signal matrices $\boldsymbol{G}_{1i}$ and $\boldsymbol{G}_{2i}$ and the system matrix $\boldsymbol{L}_{oi}$.

Two examples of first-order optical systems with adapted signals might be instructive. Let us first consider a Fourier-transform-type system for which $\boldsymbol{A} = \boldsymbol{D} = \mathbf{0}$ and $\boldsymbol{C}^t = -\boldsymbol{B}^{-1}$. Adaptation then occurs for $\boldsymbol{H}_i = \boldsymbol{H}_o = \mathbf{0}$, i.e., for Gaussian beams with zero curvature. Since the system now performs a Fourier transformation on a real, Gaussian input signal, the output signal will be real Gaussian, as well, and the output matrices $\boldsymbol{G}_o$ are proportional to the inverses of the input matrices $\boldsymbol{G}_i$, in conformity with Eqs. (30).

As a second example we consider a free-space-type system for which $\boldsymbol{A} = \boldsymbol{D} = \mathbf{I}$, $\boldsymbol{B} = \boldsymbol{B}^t$, and $\boldsymbol{C} = \mathbf{0}$. Such a system can be modeled as a Fourier-transform-type kernel (see above, with the additional condition $\boldsymbol{B} = \boldsymbol{B}^t$) preceded and succeeded by phase-compensating concave-lens-type systems:

$$\begin{bmatrix}\mathbf{I} & \boldsymbol{B}\\ \mathbf{0} & \mathbf{I}\end{bmatrix} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \boldsymbol{B}^{-1} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{0} & \boldsymbol{B}\\ -\boldsymbol{B}^{-1} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \boldsymbol{B}^{-1} & \mathbf{I}\end{bmatrix}.$$

Adaptation now occurs for $\boldsymbol{H}_i = -\boldsymbol{H}_o = \boldsymbol{B}^{-1}$, which implies that the input beam is converging and the output beam diverging. The curvature $\boldsymbol{H}_i = \boldsymbol{B}^{-1}$ of the converging input beam is first compensated by the concave-lens-type system at the input; the Fourier-transform-type kernel then transforms the real, Gaussian signal into another real, Gaussian signal; and finally the concave-lens-type system at the output transforms the beam into a diverging one and takes care of the output curvature $\boldsymbol{H}_o = -\boldsymbol{B}^{-1}$.

5.3 Symplectic case: $\boldsymbol{G}_1 = \sigma^2 \boldsymbol{G}_2$ and $\boldsymbol{H} = \boldsymbol{H}^t$

In the special case that $\boldsymbol{G}_1$ is proportional to $\boldsymbol{G}_2$, a zero-twist signal satisfies the conditions (24), and we conclude that zero twist is preserved. This conclusion is in accordance with the fact that a signal for which $\boldsymbol{G}_1 = \sigma \boldsymbol{G} = \sigma^2 \boldsymbol{G}_2$ $(0 < \sigma \leq 1)$ and $\boldsymbol{H} = \boldsymbol{H}^t$, leads to a moment matrix $\boldsymbol{M}$ that is proportional to a symplectic matrix with proportionality factor σ, and that symplecticity is preserved when such a moment matrix propagates through a (symplectic) first-order optical ***ABCD*** system [13, 20]. Moreover, it has been shown [13, 20] that the proportionality factor σ is a measure of the coherence of the signal, and that the propagation of such a signal is governed by the bilinear relationship

$$\boldsymbol{H}_i \pm i\boldsymbol{G}_i = [\boldsymbol{C} + \boldsymbol{D}(\boldsymbol{H}_o \pm i\boldsymbol{G}_o)][\boldsymbol{A} + \boldsymbol{B}(\boldsymbol{H}_o \pm i\boldsymbol{G}_o)]^{-1} \tag{31}$$

or, equivalently,

$$\boldsymbol{H}_o \pm i\boldsymbol{G}_o = -[\boldsymbol{D} - (\boldsymbol{H}_i \pm i\boldsymbol{G}_i)\boldsymbol{B}]^{-1}[\boldsymbol{C} - (\boldsymbol{H}_i \pm i\boldsymbol{G}_i)\boldsymbol{A}] \tag{32}$$

and by the fact that the factor σ remains constant. In the context of this paper it is important to conclude that the (zero) twist of symplectic signals is invariant under propagation through any first-order optical system.

6 Conclusions

In this paper we have introduced a measure for the twist of Gaussian light, expressed in terms of the second-order moments of the Wigner distribution function, see Eq. (10). Using the propagation law for these second-order moments between the input plane and the output plane of a first-order optical ***ABCD*** system, see Eq. (21), we were able to express the twist in one plane in terms of moments in the other plane. These general expressions [see Eqs. (22) and (23)] show that the twist in one plane is determined not only by the twist in the other plane, but also by other combinations of the moments.

Starting from the general relationships (22) and (23) and focussing on the special cases in which zero twist is preserved, see Eqs. (24), we were able to conclude the following.

- For the special first-order optical ***ABCD*** system for which $\boldsymbol{B} = \boldsymbol{0}$ – i.e. if we consider propagation of light between conjugate planes – the twist in one plane is determined by the twist in the other, and not by other combinations of the moments, see Eqs. (25). Moreover, the relationship between the twists is linear, and a zero-twist signal in one plane thus corresponds to a zero-twist signal in the other plane. Furthermore we concluded that $\boldsymbol{G}_{1i}$ and $\boldsymbol{G}_{1o}$ are related to each other in a simple way via the system matrix $\boldsymbol{A}$, and so are $\boldsymbol{G}_{2i}$ and $\boldsymbol{G}_{2o}$, see Eqs. (25).
- For a general first-order optical ***ABCD*** system, zero twist is not preserved. For any first-order optical system (with the mild condition $\det \boldsymbol{B} \neq \boldsymbol{0}$) there does exist, however, a zero-twist signal that is adapted to the system in the sense that it corresponds to a zero-twist signal in the other plane. The adaptation only affects the matrix $\boldsymbol{H}$; the matrices $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are not affected, see Eqs. (26) and (27). Furthermore we concluded that $\boldsymbol{G}_{1i}$ and $\boldsymbol{G}_{2o}^{-1}$ are related to each other in a simple way via the system matrix $\boldsymbol{B}$, and so are $\boldsymbol{G}_{2i}$ and $\boldsymbol{G}_{1o}^{-1}$, see Eqs. (30).

- Zero twist is preserved if we put some additional requirements to the matrices $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$; in particular we require that the moment matrix is symplectic. In that case zero twist is preserved (as is symplecticity of the moment matrix, in general) in any first-order optical system, and the relationship between the input and the output matrices is now elegantly described by a bilinear relationship, see Eqs. (31) and (32).

  Although symplecticity might seem rather special, we should realize that symplectic signals form a large subclass of Gaussian light [13]. Symplecticity applies, for instance, in (i) the completely coherent case, (ii) the (partially coherent) one-dimensional case, and (iii) the (partially coherent) rotationally symmetric case.

Appendix A. Positive definiteness of the moment matrix M

We can easily prove the positive definiteness of the moment matrix $\boldsymbol{M}$ with the help of the input-output relationship (19). We therefore choose an $\boldsymbol{ABCD}$ matrix as follows: (i) the matrix entries a_{11}, a_{12}, b_{11}, and b_{12} are chosen arbitrarily, and are combined into the 4-dimensional column vector $\boldsymbol{t} = [a_{11}, a_{12}, b_{11}, b_{12}]^t$; (ii) the matrix entries a_{21} and b_{21} are chosen equal to zero; (iii) the matrix entries a_{22} and b_{22} are chosen such that $a_{12}b_{22} = a_{22}b_{12}$; (iv) $\boldsymbol{C} = \boldsymbol{0}$; and (v) $\boldsymbol{D} = (\boldsymbol{A}^t)^{-1}$. It can easily be seen that such an $\boldsymbol{ABCD}$ matrix is symplectic and, as a consequence, can be interpreted as the ray transformation matrix of a physically realizable first-order optical system. We now consider the upper left entry of the matrix $\boldsymbol{M}_i$ in the left hand side of Eq. (19). This entry, being on the main diagonal of $\boldsymbol{M}_i$ and thus representing the square of an effective width, is positive. On the other hand, this entry equals $\boldsymbol{t}^t\boldsymbol{M}_o\boldsymbol{t}$, where the vector $\boldsymbol{t}$ can be chosen arbitrarily. We thus conclude that the quadratic form $\boldsymbol{t}^t\boldsymbol{M}_o\boldsymbol{t}$ is positive for any vector $\boldsymbol{t}$, with which we have proved that the real symmetric moment matrix $\boldsymbol{M}$ is positive definite.

Appendix B. Derivation of Eq. (22)

We substitute from Eq. (21) into $\boldsymbol{P}_i\boldsymbol{R}_i - \boldsymbol{R}_i\boldsymbol{P}_i^t$ [cf. Eq. (10)] and get

$$\begin{aligned}
\boldsymbol{P}_i\boldsymbol{R}_i - \boldsymbol{R}_i\boldsymbol{P}_i^t = {} & \boldsymbol{AR}_o(\boldsymbol{C}^t\boldsymbol{A} - \boldsymbol{A}^t\boldsymbol{C})\boldsymbol{R}_o\boldsymbol{A}^t + \boldsymbol{AR}_o(\boldsymbol{C}^t\boldsymbol{A} - \boldsymbol{A}^t\boldsymbol{C})\boldsymbol{P}_o\boldsymbol{B}^t \\
& + \boldsymbol{AR}_o(\boldsymbol{C}^t\boldsymbol{B} - \boldsymbol{A}^t\boldsymbol{D})\boldsymbol{P}_o^t\boldsymbol{A}^t + \boldsymbol{AR}_o(\boldsymbol{C}^t\boldsymbol{B} - \boldsymbol{A}^t\boldsymbol{D})\boldsymbol{Q}_o\boldsymbol{B}^t \\
& + \boldsymbol{AP}_o(\boldsymbol{D}^t\boldsymbol{A} - \boldsymbol{B}^t\boldsymbol{C})\boldsymbol{R}_o\boldsymbol{A}^t + \boldsymbol{AP}_o(\boldsymbol{D}^t\boldsymbol{A} - \boldsymbol{B}^t\boldsymbol{C})\boldsymbol{P}_o\boldsymbol{B}^t \\
& + \boldsymbol{AP}_o(\boldsymbol{D}^t\boldsymbol{B} - \boldsymbol{B}^t\boldsymbol{D})\boldsymbol{P}_o^t\boldsymbol{A}^t + \boldsymbol{AP}_o(\boldsymbol{D}^t\boldsymbol{B} - \boldsymbol{B}^t\boldsymbol{D})\boldsymbol{Q}_o\boldsymbol{B}^t \\
& + \boldsymbol{BP}_o^t(\boldsymbol{C}^t\boldsymbol{A} - \boldsymbol{A}^t\boldsymbol{C})\boldsymbol{R}_o\boldsymbol{A}^t + \boldsymbol{BP}_o^t(\boldsymbol{C}^t\boldsymbol{A} - \boldsymbol{A}^t\boldsymbol{C})\boldsymbol{P}_o\boldsymbol{B}^t \\
& + \boldsymbol{BP}_o^t(\boldsymbol{C}^t\boldsymbol{B} - \boldsymbol{A}^t\boldsymbol{D})\boldsymbol{P}_o^t\boldsymbol{A}^t + \boldsymbol{BP}_o^t(\boldsymbol{C}^t\boldsymbol{B} - \boldsymbol{A}^t\boldsymbol{D})\boldsymbol{Q}_o\boldsymbol{B}^t \\
& + \boldsymbol{BQ}_o(\boldsymbol{D}^t\boldsymbol{A} - \boldsymbol{B}^t\boldsymbol{C})\boldsymbol{R}_o\boldsymbol{A}^t + \boldsymbol{BQ}_o(\boldsymbol{D}^t\boldsymbol{A} - \boldsymbol{B}^t\boldsymbol{C})\boldsymbol{P}_o\boldsymbol{B}^t \\
& + \boldsymbol{BQ}_o(\boldsymbol{D}^t\boldsymbol{B} - \boldsymbol{B}^t\boldsymbol{D})\boldsymbol{P}_o^t\boldsymbol{A}^t + \boldsymbol{BQ}_o(\boldsymbol{D}^t\boldsymbol{B} - \boldsymbol{B}^t\boldsymbol{D})\boldsymbol{Q}_o\boldsymbol{B}^t.
\end{aligned}$$

After substituting from the symplecticity conditions (13) and (14), the latter expression reduces to

$$\begin{aligned}
\boldsymbol{P}_i\boldsymbol{R}_i - \boldsymbol{R}_i\boldsymbol{P}_i^t = {} & (\boldsymbol{AP}_o\boldsymbol{R}_o\boldsymbol{A}^t - \boldsymbol{AR}_o\boldsymbol{P}_o^t\boldsymbol{A}^t) + (\boldsymbol{AP}_o\boldsymbol{P}_o\boldsymbol{B}^t - \boldsymbol{BP}_o^t\boldsymbol{P}_o^t\boldsymbol{A}^t) \\
& + (\boldsymbol{BQ}_o\boldsymbol{R}_o\boldsymbol{A}^t - \boldsymbol{AR}_o\boldsymbol{Q}_o\boldsymbol{B}^t) + (\boldsymbol{BQ}_o\boldsymbol{P}_o\boldsymbol{B}^t - \boldsymbol{BP}_o^t\boldsymbol{Q}_o\boldsymbol{B}^t).
\end{aligned}$$

We now substitute from the relations $\boldsymbol{R}_o = \frac{1}{2}\boldsymbol{G}_1^{-1}$, $\boldsymbol{P}_o = \frac{1}{2}\boldsymbol{G}_1^{-1}\boldsymbol{H}$, and $\boldsymbol{Q}_o = \frac{1}{2}(\boldsymbol{G}_2 + \boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{H})$ [see Eqs. (7)], after which the latter expression takes the form

$$\begin{aligned}4(\boldsymbol{P}_i\boldsymbol{R}_i - \boldsymbol{R}_i\boldsymbol{P}_i^t) \;=\;& \boldsymbol{A}\boldsymbol{G}_1^{-1}(\boldsymbol{H}-\boldsymbol{H}^t)\boldsymbol{G}_1^{-1}\boldsymbol{A}^t \\ &+ (\boldsymbol{A}\boldsymbol{G}_1^{-1}\boldsymbol{H}\boldsymbol{G}_1^{-1}\boldsymbol{H}\boldsymbol{B}^t - \boldsymbol{B}\boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{A}^t) \\ &+ (\boldsymbol{B}\boldsymbol{G}_2\boldsymbol{G}_1^{-1}\boldsymbol{A}^t - \boldsymbol{A}\boldsymbol{G}_1^{-1}\boldsymbol{G}_2\boldsymbol{B}^t) \\ &+ (\boldsymbol{B}\boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{H}\boldsymbol{G}_1^{-1}\boldsymbol{A}^t - \boldsymbol{A}\boldsymbol{G}_1^{-1}\boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{H}\boldsymbol{B}^t) \\ &+ (\boldsymbol{B}\boldsymbol{G}_2\boldsymbol{G}_1^{-1}\boldsymbol{H}\boldsymbol{B}^t - \boldsymbol{B}\boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{B}^t) \\ &+ \boldsymbol{B}\boldsymbol{H}^t\boldsymbol{G}_1^{-1}(\boldsymbol{H}-\boldsymbol{H}^t)\boldsymbol{G}_1^{-1}\boldsymbol{H}\boldsymbol{B}^t\end{aligned}$$

We finally combine terms to get Eq. (22)

$$\begin{aligned}4(\boldsymbol{P}_i\boldsymbol{R}_i - \boldsymbol{R}_i\boldsymbol{P}_i^t) \;=\;& (\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}^t)\boldsymbol{G}_1^{-1}(\boldsymbol{H}-\boldsymbol{H}^t)\boldsymbol{G}_1^{-1}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}^t)^t \\ &- (\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}^t)\boldsymbol{G}_1^{-1}\boldsymbol{G}_2\boldsymbol{B}^t + \boldsymbol{B}\boldsymbol{G}_2\boldsymbol{G}_1^{-1}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}^t)^t.\end{aligned}$$

The author can be reached at the address on the title page, by telephone at +31 40 2473319, by fax at +31 40 2448375, or by e-mail at M.J.Bastiaans@ele.tue.nl.

References

[1] R. Simon and N. Mukunda, “Twisted Gaussian Schell-model beams,” J. Opt. Soc. Am. A **10**, 95-109 (1993).

[2] R. Simon, K. Sundar, and N. Mukunda, “Twisted Gaussian Schell-model beams. I. Symmetry structure and normal-mode spectrum,” J. Opt. Soc. Am. A **10**, 2008-2016 (1993).

[3] K. Sundar, R. Simon, and N. Mukunda, “Twisted Gaussian Schell-model beams. II. Spectrum analysis and propagation characteristics,” J. Opt. Soc. Am. A **10**, 2017-2023 (1993).

[4] A. T. Friberg, B. Tervonen, and J. Turunen, “Interpretation and experimental demonstration of twisted Gaussian Schell-model beams,” J. Opt. Soc. Am. A **11**, 1818-1826 (1994).

[5] D. Ambrosini, V. Bagini, F. Gori, and M. Santarsiero, “Twisted Gaussian Schell-model beams: a superposition model,” J. Mod. Opt. **41**, 1391-1399 (1994).

[6] R. Simon, A. T. Friberg, and E. Wolf, “Transfer of radiance by twisted Gaussian Schell-model beams in paraxial systems,” J. Eur. Opt. Soc. A: Pure Appl. Opt. **5**, 331-343 (1996).

[7] R. Simon and N. Mukunda, “Twist phase in Gaussian-beam optics,” J. Opt. Soc. Am. A, **15**, 2373 (1998).

[8] M. J. Bastiaans, “On the propagation of the twist of Gaussian light in first-order optical systems,” O. P. Nijhawan, A. K. Gupta, A. K. Musla, and Kehar Singh, eds. *Optics and Optoelectronics, Theory, Devices and Applications* (Narosa Publishing House, New Delhi, India, 1998), pp. 121-125.

[9] M. J. Bastiaans, “Zero twist of Gaussian light in first-order optical systems,” *Optics for the Next Millennium*, SPIE Vol. **3749** (SPIE – The International Society for Optical Engineering, Bellingham, WA, 1999), pp. 112-113.

[10] A. Papoulis, *Systems and Transforms with Applications in Optics* (McGraw-Hill, New York, 1968).

[11] L. Mandel and E. Wolf, "Spectral coherence and the concept of cross-spectral purity," J. Opt. Soc. Am. **66**, 529-535 (1976).
[12] M. J. Bastiaans, "A frequency-domain treatment of partial coherence," Opt. Acta **24**, 261-274 (1977).
[13] M. J. Bastiaans, "Wigner distribution function applied to partially coherent light," in *Proceedings of the Workshop on Laser Beam Characterization*, P. M. Mejías, H. Weber, R. Martínez-Herrero, and A. González-Ureña, eds. (Sociedad Española de Optica, Madrid, 1993), pp. 65-87.
[14] A. C. Schell, "A technique for the determination of the radiation pattern of a partially coherent aperture," IEEE Trans. Antennas Propag. **AP-15**, 187-188 (1967).
[15] F. Gori, "Collett-Wolf sources and multimode lasers," Opt. Commun. **34**, 301-305, (1980).
[16] E. Wigner, "On the quantum correction for thermodynamic equilibrium," Phys. Rev. **40**, 749-759 (1932).
[17] A. Walther, "Radiometry and coherence," J. Opt. Soc. Am. **58** 1256-1259 (1968).
[18] M. J. Bastiaans, "Application of the Wigner distribution function to partially coherent light," J. Opt. Soc. Am. A **3**, 1227-1238 (1986).
[19] R. Simon, E. C. G. Sudarshan, and N. Mukunda, "Anisotropic Gaussian Schell-model beams: Passage through optical systems and associated invariants," Phys. Rev. A **31**, 2419-2434 (1985).
[20] M. J. Bastiaans, "ABCD law for partially coherent Gaussian light, propagating through first-order optical systems," Opt. Quant. Electron. **24**, 1011-1019 (1992).
[21] R. Simon, E. C. G. Sudarshan, and N. Mukunda, "Gaussian-Wigner distributions in quantum mechanics and optics," Phys. Rev. A **36**, 3868-3880 (1987).
[22] R. Simon, N. Mukunda, and B. Dutta, "Quantum-noise matrix for multimode systems – U(n) invariance, squeezing, and normal forms," Phys. Rev. A **49**, 1567-1583 (1994).
[23] R. K. Luneburg, *Mathematical Theory of Optics* (Univ. California Press, Berkeley, 1966).
[24] G. A. Deschamps, "Ray techniques in electromagnetics," Proc. IEEE **60**, 1022-1035 (1972).
[25] M. J. Bastiaans, "Wigner distribution function and its applications to first-order optics," J. Opt. Soc. Am. **69**, 1710-1716 (1979).
[26] R. Simon, N. Mukunda, and E. C. G. Sudarshan, "Partially coherent beams and a generalized ABCD-law," Opt. Commun. **65**, 322-328 (1988).
[27] M. J. Bastiaans, "Second-order moments of the Wigner distribution function in first-order optical systems," Optik **88**, 163-168 (1991).
[28] J. W. Goodman, *Introduction to Fourier Optics: Second Edition* (McGraw-Hill, New York, 1996).