

# Evolution of the vortex and asymmetrical parts of the orbital angular momentum in separable first-order optical systems

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We analyze the evolution of the vortex and asymmetrical parts of the orbital angular momentum during its propagation through separable first-order optical systems. We obtain that the evolution of the vortex part depends only upon the parameters  $a_x$ ,  $a_y$ ,  $b_x$ , and  $b_y$  of the ray transformation matrix, and that isotropic systems with the same ratio  $b/a$  produce the same change of the vortex part of the orbital angular momentum. Finally, it is shown that when light propagates through an optical fiber with a quadratic refractive index profile, the vortex part of the orbital angular momentum cannot change its sign more than four times per period.

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OCIS codes: 070.2580, 070.4690, 030.5630, 120.4820, 140.3300

During the last decade the concept of the orbital angular momentum (OAM) has been applied for the description of coherent optical vortex beams.<sup>1,2</sup> In recent publications<sup>3,4</sup> it has been suggested to decompose the OAM into two parts: the asymmetrical OAM and the vortex OAM. The first part describes an astigmatic beam but with a smooth wave front, while the second one is related to the singularity of the wave front (screw dislocations).

In this contribution we study the evolution of the asymmetrical OAM and vortex OAM of linearly polarized, partially coherent beams during their propagation through separable first-order optical – also called *ABCD* – systems. It has recently been reported that some partially coherent fields also exhibit the vortex behavior.<sup>5</sup> Taking into account that the concept of the OAM can be generalized to the case of partially coherent beams,<sup>6</sup> the results of this study can be applied to both the completely coherent and the partially coherent case.

We will base the definition of the OAM on the moments of the Wigner distribution<sup>7</sup> (WD). The WD  $W(x, u; y, v)$  represents partially coherent light in a combined space/spatial-frequency domain, the so-called phase space, where  $u$  is the spatial-frequency variable associated to the space variable  $x$ , and  $v$  the spatial-frequency variable associated to the space variable  $y$ . The treatment in this paper is based on the normalized moments of the Wigner distribution (WD), where the normalization is with respect to the total energy  $E$  of the signal:  $E = \iiint W(x, u; y, v) dx du dy dv$ . These normalized moments  $\mu_{pqrs}$  of the WD are thus defined by

$$\mu_{pqrs}E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(x, u; y, v) x^p u^q y^r v^s dx du dy dv \quad (p, q, r, s \geq 0). \quad (1)$$

In the present paper we restrict ourselves to the ten second-order moments,  $p+q+r+s=2$ , assuming moreover, without loss of generality, that the four first-order moments are zero.

The OAM  $\Lambda$  of an optical beam can be expressed in terms of the second-order moments of its WD as [4, Eq. (3)]  $\Lambda = (E/c^2)(\mu_{1001} - \mu_{0110})$ . The asymmetrical part  $\Lambda_a$  and the vortex part  $\Lambda_v$  of the OAM, can then be written as [4, Eqs. (22) and (21)]

$$\Lambda_a = \frac{E}{c^2} \frac{(\mu_{2000} - \mu_{0020})(\mu_{1001} + \mu_{0110}) - 2\mu_{1010}(\mu_{1100} - \mu_{0011})}{\mu_{2000} + \mu_{0020}} \quad (2)$$

$$\Lambda_v = \frac{2E}{c^2} \frac{\mu_{0020}\mu_{1001} - \mu_{2000}\mu_{0110} + \mu_{1010}(\mu_{1100} - \mu_{0011})}{\mu_{2000} + \mu_{0020}}, \quad (3)$$

respectively; note that we have  $\Lambda_a + \Lambda_v = \Lambda$ . It is easy to see that for rotationally symmetric beams, for which  $\mu_{2000} = \mu_{0020}$  and  $\mu_{1010} = 0$ ,<sup>8</sup> the asymmetrical part  $\Lambda_a = 0$ . We remark that the vortex part  $\Lambda_v$  is closely related to the optical twist<sup>9</sup>  $T$ , defined as

$$T = \mu_{0020}\mu_{1001} - \mu_{2000}\mu_{0110} + \mu_{1010}(\mu_{1100} - \mu_{0011}). \quad (4)$$

To study the evolution of the WD moments in separable first-order systems, we recall that the input-output relationship between the WD  $W_{\text{in}}(x, u; y, v)$  at the input plane and the WD  $W_{\text{out}}(x, u; y, v)$  at the output plane of a separable first-order optical system reads<sup>10,11</sup>  $W_{\text{out}}(x, u; y, v) = W_{\text{in}}(d_x x - b_x u, -c_x x + a_x u; d_y y - b_y v, -c_y y + a_y v)$ , which relationship is based on an affine transformation of the WD described by the so-called ray transformation matrix<sup>12</sup> or  $ABCD$ -matrix. From the symplecticity condition<sup>12</sup> of this ray transformation matrix we have, in the special case of a separable system,  $a_x d_x - b_x c_x = 1$  and  $a_y d_y - b_y c_y = 1$ .

The normalized moments  $\mu_{pqrs}^{\text{out}}$  of the output WD  $W_{\text{out}}(x, u; y, v)$  are related to the normalized moments  $\mu_{pqrs}^{\text{in}} = \mu_{pqrs}$  of the input WD  $W_{\text{in}}(x, u; y, v)$  as

$$\begin{aligned} \mu_{pqrs}^{\text{out}} E &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{\text{in}}(x, u; y, v) (a_x x + b_x u)^p (c_x x + d_x u)^q \\ &\quad \times (a_y y + b_y v)^r (c_y y + d_y v)^s dx du dy dv \\ &= E \sum_{k=0}^p \sum_{l=0}^q \sum_{m=0}^r \sum_{n=0}^s \binom{p}{k} \binom{q}{l} \binom{r}{m} \binom{s}{n} a_x^{p-k} b_x^k c_x^l d_x^{q-l} \\ &\quad \times a_y^{r-m} b_y^m c_y^n d_y^{s-n} \mu_{p-k+l, q-l+k, r-m+n, s-n+m}. \end{aligned} \quad (5)$$

We remark that for Fourier transforming systems, with respect to one variable or to two variables, the output moments are related to the input moments by the simple relationships

$$\begin{aligned} \mu_{pqrs}^{\text{out}} |_{FT_x} &= \mu_{qprs} (-1)^q b_x^{p-q}, \\ \mu_{pqrs}^{\text{out}} |_{FT_y} &= \mu_{pqsr} (-1)^s b_y^{r-s}, \\ \mu_{pqrs}^{\text{out}} |_{FT_{xy}} &= \mu_{qpsr} (-1)^{q+s} b_x^{p-q} b_y^{r-s}, \end{aligned} \quad (6)$$

where  $FT_x$ ,  $FT_y$ , and  $FT_{xy}$  represent the Fourier transforms with respect to  $x$  (with parameters  $a_x = d_x = 0$ ,  $b_x c_x = -1$ ,  $a_y = d_y = 1$ ,  $b_y = c_y = 0$ ), to  $y$  (with parameters  $a_x = d_x = 1$ ,  $b_x = c_x = 0$ ,  $a_y = d_y = 0$ ,  $b_y c_y = -1$ ), and to both  $x$  and  $y$  (with parameters  $a_x = d_x = 0$ ,  $b_x c_x = -1$ ,  $a_y = d_y = 0$ ,  $b_y c_y = -1$ ), respectively. If we combine these simple relationships (6) with the definition of the twist (4), we can readily derive the twists  $T^{\text{out}}|_{FT_x} = b_x T_x$ ,  $T^{\text{out}}|_{FT_y} = b_y T_y$ , and  $T^{\text{out}}|_{FT_{xy}} = b_x b_y T_{xy}$  at the output of these respective Fourier transforming systems, and we have

$$\begin{aligned} T_x &= \mu_{0020}\mu_{0101} + \mu_{0200}\mu_{1010} - \mu_{0110}(\mu_{1100} + \mu_{0011}), \\ T_y &= -\mu_{0002}\mu_{1010} - \mu_{2000}\mu_{0101} + \mu_{1001}(\mu_{1100} + \mu_{0011}), \\ T_{xy} &= -\mu_{0002}\mu_{0110} + \mu_{0200}\mu_{1001} - \mu_{0101}(\mu_{1100} - \mu_{0011}). \end{aligned} \quad (7)$$

To study in particular the evolution of the twist and the orbital angular momentum, we first observe that there are seven relevant second-order moments which compose the vortex (2) and the asymmetrical (3) part of the OAM; the purely  $uv$ -moments (i.e.,  $\mu_{0200}$ ,  $\mu_{0002}$ , and  $\mu_{0101}$ , for which  $p = r = 0$ ) do not enter the formulas (2) and (3). The moment relationship (5) immediately leads to the necessary expressions that describe the values of these seven moments in the output of an  $ABCD$  system in terms of the ten second-order input moments. In particular, the OAM  $\Lambda^{\text{out}} = (E/c^2)(\mu_{1001}^{\text{out}} - \mu_{0110}^{\text{out}})$  at the output of a separable  $ABCD$  system follows from the equations

$$\begin{aligned}\mu_{1001}^{\text{out}} &= a_x c_y \mu_{1010} + a_x d_y \mu_{1001} + b_x c_y \mu_{0110} + b_x d_y \mu_{0101}, \\ \mu_{0110}^{\text{out}} &= c_x a_y \mu_{1010} + c_x b_y \mu_{1001} + d_x a_y \mu_{0110} + d_x b_y \mu_{0101},\end{aligned}$$

and takes the form

$$\begin{aligned}\Lambda^{\text{out}} &= \frac{E}{c^2} [\mu_{1001}(a_x d_y - b_y c_x) - \mu_{0110}(a_y d_x - b_x c_y) \\ &\quad + \mu_{0101}(b_x d_y - b_y d_x) + \mu_{1010}(a_x c_y - a_y c_x)].\end{aligned}\quad (8)$$

In order to obtain the evolution of the vortex and asymmetrical parts of the OAM, let us first consider the output twist  $T^{\text{out}}$ . After a straightforward but rather lengthy calculation, using the definition of the twist (4), the moment relations (5), and the definitions (7), it can be shown that the following expression holds for the output twist:

$$T^{\text{out}} = a_x a_y T + a_y b_x T_x + a_x b_y T_y + b_x b_y T_{xy}.\quad (9)$$

We further need the expression

$$\mu_{2000}^{\text{out}} + \mu_{0020}^{\text{out}} = a_x^2 \mu_{2000} + 2a_x b_x \mu_{1100} + b_x^2 \mu_{0200} + a_y^2 \mu_{0020} + 2a_y b_y \mu_{0011} + b_y^2 \mu_{0002}\quad (10)$$

for the denominator that appears in Eqs. (2) and (3). Note that the denominator is always positive, which is a direct consequence of the fact that the moments  $\mu_{2000}$  and  $\mu_{0020}$  (and also  $\mu_{0200}$  and  $\mu_{0002}$ ) are positive by definition. From Eqs. (9) and (10), we immediately conclude that the evolution of the vortex part  $\Lambda_v$  through a separable  $ABCD$  system depends only on the parameters  $a_x$ ,  $a_y$ ,  $b_x$ , and  $b_y$  of the ray transformation matrix. This also implies that  $\Lambda_v$  cannot be changed by a quadratic phase corrector<sup>3,4</sup> ( $b_x = b_y = 0$ ,  $a_x = a_y = 1$ ); note that in this case  $\Lambda^{\text{out}} = \Lambda + (E/c^2)\mu_{1010}(c_y - c_x)$ .

For an isotropic system, i.e.,  $a_x = a_y$ ,  $b_x = b_y$ ,  $c_x = c_y$ , and  $d_x = d_y$ , the general equation (8) reduces to  $\Lambda^{\text{out}} = (E/c^2)(\mu_{1001} - \mu_{0110}) = \Lambda$  and we conclude that the OAM does not change when the beam propagates through an isotropic  $ABCD$  system. However, this property does not hold for the vortex and asymmetrical parts of the OAM, as has been demonstrated for the case of free-space propagation.<sup>3,4</sup> In the special case of an isotropic system, Eqs. (9) and (10) reduce to simpler forms and we get

$$\Lambda_v^{\text{out}} = \frac{2E}{c^2} \frac{a^2 T + ab(T_x + T_y) + b^2 T_{xy}}{a^2(\mu_{2000} + \mu_{0020}) + 2ab(\mu_{1100} + \mu_{0011}) + b^2(\mu_{0200} + \mu_{0002})}.\quad (11)$$

We remark that for the special signal for which the relation

$$\frac{T}{\mu_{2000} + \mu_{0020}} = \frac{T_x + T_y}{2(\mu_{1100} + \mu_{0011})} = \frac{T_{xy}}{\mu_{0200} + \mu_{0002}}\quad (12)$$

holds, the vortex part of its OAM does not change in isotropic systems. This holds in particular for rotationally symmetric beams, for which  $\mu_{0020} = \mu_{2000}$ ,  $\mu_{0002} = \mu_{0200}$ ,  $\mu_{0011} = \mu_{1100}$ ,  $\mu_{0110} = -\mu_{1001}$ , and  $\mu_{1010} = \mu_{0101} = 0$ ,<sup>8</sup> and hence  $T = 2\mu_{2000}\mu_{1001}$ ,  $T_x = T_y = 2\mu_{1100}\mu_{1001}$ , and  $T_{xy} = 2\mu_{0200}\mu_{1001}$ , and for which the three expressions in Eq. (12) thus take the value  $\mu_{1001}$ : a rotationally symmetric beam propagating through an isotropic system does not change its vortex and the asymmetrical part of its OAM remains zero.

We note that for isotropic systems with  $b = 0$  (and  $ad = 1$ ), the vortex part of the OAM (as well as the asymmetrical part of it) is preserved:  $\Lambda_v^{\text{out}} = \Lambda_v$  and  $\Lambda_a^{\text{out}} = \Lambda_a$ . This is the case for all systems for which the input and output planes are conjugate planes.

Let us now study the general expression (11) for the evolution of the vortex part of the OAM, which we may as well express in the form

$$\Lambda_v^{\text{out}} = \Lambda_v(p) = \frac{2E}{c^2} \frac{T + p(T_x + T_y) + p^2 T_{xy}}{(\mu_{2000} + \mu_{0020}) + 2p(\mu_{1100} + \mu_{0011}) + p^2(\mu_{0200} + \mu_{0002})}, \quad (13)$$

where we have introduced the ratio  $p = b/a$ . Optical systems with the same  $p$  behave similarly with respect to the evolution of  $\Lambda_v$ . As examples we mention (i) a section of free space in the paraxial approximation, or ‘parabolic’ system<sup>13</sup> (with  $a = d = 1$ ,  $c = 0$ , and  $b$  proportional to the propagation distance), (ii) a fractional Fourier transforming system,<sup>14</sup> or ‘elliptic’ system<sup>13</sup> (with  $a = d = \cos \alpha$  and  $b = -c = \sin \alpha$ ), and (iii) a ‘hyperbolic’ system<sup>13</sup> (with  $a = d = \cosh \alpha$  and  $b = c = \sinh \alpha$ ), for which systems the parameter  $p$  corresponds to  $b$ ,  $\tan \alpha$ , and  $\tanh \alpha$ , respectively. Analyzing Eq. (13), one can find the values of the parameter  $p$  where  $\Lambda_v(p)$  is zero, or has maxima or minima. Note that, while the denominator is positive for all possible values of  $p$ , the numerator – and therefore  $\Lambda_v(p)$  itself – may or may not change its sign (but not more than twice) depending upon the actual values of  $T$ ,  $T_x + T_y$ , and  $T_{xy}$ . Clearly,  $\Lambda_v(p)$  takes the value zero for those values of the parameter  $p$  for which  $p = \{-(T_x + T_y) \pm [(T_x + T_y)^2 - 4TT_{xy}]^{1/2}\}/2T_{xy}$ , and since  $p$  has to be real, this is possible only if the condition  $(T_x + T_y)^2 \geq 4TT_{xy}$  holds.

In the case of free-space propagation, for which  $p$  is associated with the propagation distance, only positive values of  $p$  are allowed, and  $\Lambda_v(0) = (2E/c^2)T/(\mu_{2000} + \mu_{0020})$  and  $\Lambda_v(p \rightarrow \infty) = (2E/c^2)T_{xy}/(\mu_{0200} + \mu_{0002})$ ; note that the case  $p \rightarrow \infty$  corresponds to Fourier transforming. Hence, for  $TT_{xy} < 0$ , there is only *one* positive solution where  $\Lambda(p)$  changes its sign. On the other hand, for  $TT_{xy} > 0$ , there are *two* positive solutions if the signs of  $T_{xy}$  and  $T_x + T_y$  are different, and there are *no* positive solutions if these signs are the same. Moreover, in the special case that the initial field (or its Fourier transform) is vortex free,  $T = 0$  (or  $T_{xy} = 0$ ), the vortex  $\Lambda_v(p)$  takes the value zero also for  $p = -(T_x + T_y)/T_{xy}$ , which is positive if  $T_{xy}$  and  $T_x + T_y$  have different signs (or for  $p = -T/(T_x + T_y)$ , which is positive if  $T$  and  $T_x + T_y$  have different signs).

In the case of a fractional Fourier transforming system, for which  $p$  is associated with the fractional angle  $\alpha$ ,  $p = \tan \alpha$ , both positive and negative values of  $p$  are acceptable. Positive values of  $p$  correspond to the first and third quadrants of the angle  $\alpha$ , and negative values to the second and fourth ones. Therefore, during one period  $0 \leq \alpha \leq 2\pi$ , the sign of  $\Lambda_v(p)$  changes four times if  $(T_x + T_y)^2 > 4TT_{xy}$ , twice if  $(T_x + T_y)^2 = 4TT_{xy}$ , and it does not change if  $(T_x + T_y)^2 < 4TT_{xy}$ . Note that this analysis is useful for understanding the propagation of vortex beams through optical fibers with quadratic refractive index profiles, since in the paraxial approximation of the scalar diffraction theory, the complex field amplitude of monochromatic light propagating through such a fiber is fractionally Fourier transformed, with the angle  $\alpha$  being proportional to the propagation distance in the fiber.<sup>15,16</sup>

## Acknowledgment

T. Alieva thanks the Spanish Ministry of Science and Technology for financial support of this research (“Ramon y Cajal” grant and project TIC 2002-01846).

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