WIGNER DISTRIBUTION RECONSTRUCTION FROM TWO PROJECTIONS

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ABSTRACT

The connection between the instantaneous frequency and the angle derivative of the fractional power spectra is established. It permits to solve the signal retrieval problem if only two close fractional power spectra are known. This fact is used in the reconstruction of the Wigner distribution or the pseudo Wigner distribution from two close projections.

1. INTRODUCTION

The reconstruction of a signal – and in particular its phase – from the distributions associated with the instantaneous power of the signal, its power spectrum or, more general, its fractional power spectra, is an important problem in signal processing, radio location, optics, quantum mechanics, etc. In spite of several successful iterative algorithms for phase reconstruction from the squared modulus of the signal and its power spectrum, or its Fresnel spectrum, which were proposed recently [1]-[3], the development of noniterative procedures remains an attractive research topic.

The fractional power spectra, which are the squared moduli of the fractional Fourier transform (FT) [4], are now a popular tool in optics and signal processing [4]-[11]. As it is known, they are equal to the projections of the Wigner distribution (WD) of the signal under consideration [11, 12]. Thus, by using the tomographic approach and the inverse Radon transform, the WD – and therefore the signal itself, up to a constant phase factor – can be reconstructed by knowing all its projections [5, 8]. The method is based on the rotation in the time-frequency plane of the WD under the fractional FT. It demands the measurements of the fractional FT spectra in the wide angular region $[0, \pi)$, which sometimes is impossible or very cost consuming [5].

In this paper we propose a new approach for the WD reconstruction from only two fractional FT spectra, i.e., only two WD projections. This approach significantly reduces the need for projections measurements and calculations. Its is also direct and does not use iterative procedures.

The paper is organized as follows. In Section 2 we present a review of the definition of the fractional FT, and the relationship between the fractional FT power spectra and the ambiguity function of a signal. In Section 3 we establish the connection between the instantaneous frequency in a fractional domain and the angular derivative of the fractional FT power spectra. We show that the instantaneous frequency is determined by the convolution of the angular derivative of the fractional power spectra and the signum function. In Section 4 we discuss the discrete version of the proposed phase retrieval method and demonstrate its efficiency on examples. The importance of the new algorithm and its possible applications are discussed in the Conclusions.

2. FRACTIONAL POWER SPECTRA AND AMBIGUITY FUNCTION

The fractional FT of function $x(t)$, can be written in the form [4]

$$R^{\alpha}[x(t)](u) = X_{\alpha}(u) = \int_{-\infty}^{\infty} K(\alpha, t, u) x(t) dt, \qquad (1)$$

where the kernel $K(\alpha, t, u)$ is given by

$$K(\alpha, t, u) = \frac{\exp(j\frac{1}{2}\alpha)}{\sqrt{j \sin\alpha}} \exp(j\pi \frac{(t^2 + u^2)\cos\alpha - 2ut}{\sin\alpha}). \qquad (2)$$

Note that, in particular, $X_0(u) = x(u)$, $X_{\pi}(u) = x(-u)$, and that $X_{\pi/2}(u)$ corresponds to a normal FT. This transform is additive on the parameter α which corresponds to the rotation angle of the coordinate system.

It is known that the fractional power spectra $|X_{\alpha}(u)|^2$, i.e., the squared moduli of the fractional FT, are equal to the projections of the WD $W_x(t, f)$ of the signal $x(t)$,

$$|X_{\alpha}(u)|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_x(t, f)\delta(t\cos\alpha + f\sin\alpha - u) df dt$$
$$= \int_{-\infty}^{\infty} W_x(u\cos\alpha - f\sin\alpha, u\sin\alpha + f\cos\alpha) df. \qquad (3)$$

The set of fractional power spectra in the angular region $[0, \pi)$ is also called the Radon-Wigner transform.

Since the ambiguity function $A_x(\tau, \nu)$ is the two-dimensional FT of the WD $W_x(t, f)$, the values of the ambiguity function along the line defined by α are – according to the Radon transform properties – equal to the FT of the WD projection for the same α [6, 8],

$$A_x(R\sin\alpha, -R\cos\alpha) = \int_{-\infty}^{\infty} |X_{\alpha}(u)|^2 e^{j2\pi Ru} du. \qquad (4)$$

We can also say that the fractional power spectrum $|X_{\alpha}(u)|^2$ is the FT of the ambiguity function. Note that this relationship is very important for the experimental determination of the ambiguity function in optics, where the fractional power spectra related to intensity distributions can be measured by a simple optical setup [6].

3. WIGNER DISTRIBUTION PROJECTIONS AND INSTANTANEOUS FREQUENCIES

In this section we will derive that the well-known expression for the instantaneous frequency $f_0(t)$ at the position t [13]

$$f_0(t) = \frac{1}{2\pi j}\frac{1}{|x(t)|^2}\int_{-\infty}^{\infty}\left.\frac{\partial A_x(\tau,\nu)}{\partial\tau}\right|_{\tau=0} e^{j2\pi t\nu}d\nu, \quad (5)$$

can be written in terms of the local moments of the fractional power spectra. Indeed, using the relationship [14]

$$\left.\frac{\partial A_x(\tau,\nu)}{\partial\tau}\right|_{\tau=0} = -\frac{1}{\nu}\int_{-\infty}^{\infty}\left.\frac{\partial\,|X_\alpha(u)|^2}{\partial\alpha}\right|_{\alpha=0} e^{-j2\pi\nu u}du, \quad (6)$$

we get

$$f_0(t) = \frac{-1}{2\,|X_0(t)|^2}\int_{-\infty}^{\infty}\left.\frac{\partial\,|X_\alpha(u)|^2}{\partial\alpha}\right|_{\alpha=0}\mathrm{sgn}(t-u)du, \quad (7)$$

where $\mathrm{sgn}(t)$ is the signum function:

$$\mathrm{sgn}(t) = \frac{1}{\pi j}\int_{-\infty}^{\infty}\frac{1}{\nu}e^{j2\pi\nu t}d\nu = \begin{cases} 1 \text{ for } t>0, \\ -1 \text{ for } t<0. \end{cases} \quad (8)$$

We thus get for the signal $x(t) = |X_0(t)|\exp[j\varphi(t)]$, that its phase derivative $\varphi'(t) = d\varphi(t)/dt = 2\pi f_0(t)$ is determined by its intensity $|X_0(t)|^2$ and the convolution of the signum function with the angular derivative of the fractional power spectrum $\partial|X_\alpha(u)|^2/\partial\alpha$ at the angle $\alpha = 0$. Note that this relationship can easily be generalized for an arbitrary angle $\alpha \neq 0$, using $|X_\alpha(t)|^2$ and $f_\alpha(t)$ [14] .

In general, the complex-valued fractional FT $X_\alpha(t)$, and in particular the signal $x(t) = X_0(t)$, can be completely reconstructed (except for a constant phase shift) from its intensity distribution $|X_\alpha(t)|^2$ and its instantaneous frequency $f_\alpha(t)$. Since the instantaneous frequency is determined by the derivative of the fractional power spectra, see Eq. (7), this implies that only two fractional power spectra for close angles suffice to solve the signal retrieval problem, up to a constant phase factor. By reconstructing the signal, up to this constant phase factor, from two fractional power spectra (i.e., two WD projections), we can reconstruct the whole WD. Because $x(t)$ is related to $X_\alpha(t)$ through the inverse fractional FT, we can conclude that the signal phase can be reconstructed up to a constant term by a noniterative way from any two fractional power spectra taken for close angles.

Note that this result resembles the so-called transport of intensity equation, which deals with the Fresnel transformation [15]-[17]. This is not surprising since both the fractional FT and the Fresnel transform belong to the class of canonical integral transforms and the properties of any member of this class are related, too.

4. DISCRETIZATION AND EXAMPLES

4.1. Discretization

Here we will illustrate on some numerical examples how the signal, up to a constant phase factor, and its WD can be reconstructed from only two close fractional power spectra, i.e., two WD projections. Of course, instead of reconstructing the WD, we will actually reconstruct the *pseudo* WD, but for an appropriate window function and a small angle α, this will not have any noticeable effect on the final results.

After two WD projections have been obtained or in practice measured as fractional power spectra by using an appropriate optical setup, the instantaneous frequency is calculated as the output of the linear system, cf. Eq. (7),

$$f_0(nT) = -\frac{1}{2\alpha}\frac{|X_\alpha(nT)|^2 - |X_{-\alpha}(nT)|^2 *_n \mathrm{sgn}(nT)}{2\,|X_0(nT)|^2}T \quad (9)$$

where T is the discretization step, the angle α is small, and $*_n$ denotes the discrete-time convolution; moreover, in order to avoid a separate estimation of $|X_0(nT)|^2$, for small α the denominator $2|X_0(nT)|^2$ can be approximated by $|X_\alpha(nT)|^2 + |X_{-\alpha}(nT)|^2$. Note that instead of this symmetrical version of the system, we might as well have chosen an asymmetrical one with $-\alpha$ replaced by 0 and 2α by α.

The signal, up to the constant phase factor, is reconstructed as

$$\hat{x}(nT) = |X_0(nT)|\exp[j\sum_{m=-N}^{N}\varphi'(mT)T] \quad (10)$$

and the (pseudo) WD is calculated according to its definition

$$W_{\hat{x}}(n,k) = 2T\sum_{m=-N}^{N-1}\hat{x}[(n+m)T]\hat{x}^*[(n-m)T] \times w(mT)e^{-j2\pi mk/N}, \quad (11)$$

where $w(nT)$ is an appropriately chosen window function and where N is chosen such that $\hat{x}(nT) \approx 0$ for $|n| \geq N$.

The fractional spectra $|X_\alpha(nT)|^2$ and $|X_{-\alpha}(nT)|^2$ can be obtained in different ways: (i) measured in experiments (a simple optical set up for the measurements of the fractional power spectra was described in [18]); (ii) calculated as squared moduli of the corresponding fractional FT of $x(t)$; (iii) calculated as the Radon transform of the WD of $x(t)$ for two angles $\pm\alpha$.

4.2. Examples

Example 1: We start with the reconstruction of a **monocomponent signal**, whose instantaneous frequency is a monotonic function. Thus a signal of the form

$$x(t) = \exp[-(2.25t)^8] \times \exp\{j\int_{-\infty}^{t}[40\pi\sinh^{-1}(100t) + 256\pi t]dt\} \quad (12)$$

is considered, with $T = 1/1024$. Its (pseudo) WD is calculated, by using a Hanning window $w(t)$ with width $T_w = 1/8$. After the WD has been obtained, we assume that only two of its projections are known, for angles $\alpha = -1^\circ$ and $\alpha = 1^\circ$. The projections are calculated by using the MATLAB radon function, taking the WD matrix as the argument. This corresponds to the case where two fractional power spectra $|X_\alpha(nT)|^2$ and $|X_{-\alpha}(nT)|^2$ are obtained by measurements in an optical system. The procedure described before [cf. Eq. (9] is then used for the reconstruction

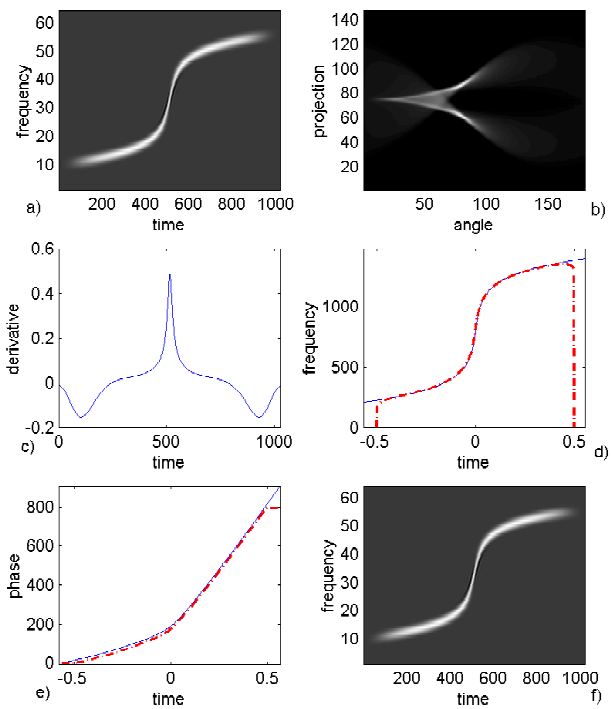

Figure 1: Monocomponent signal and its WD reconstruction from two close fractional power spectra (two WD projections): a) Original WD, b) Projections of the WD (Radon-Wigner transform), c) Derivative approximation: difference of two close projections calculated at 1° and -1°, and divided by the angle step, d) Reconstructed (dash-dot) and original (solid line) instantaneous frequency of the signal, e) Reconstructed (dash-dot) and original (solid line) phase of the signal, f) Reconstructed WD.

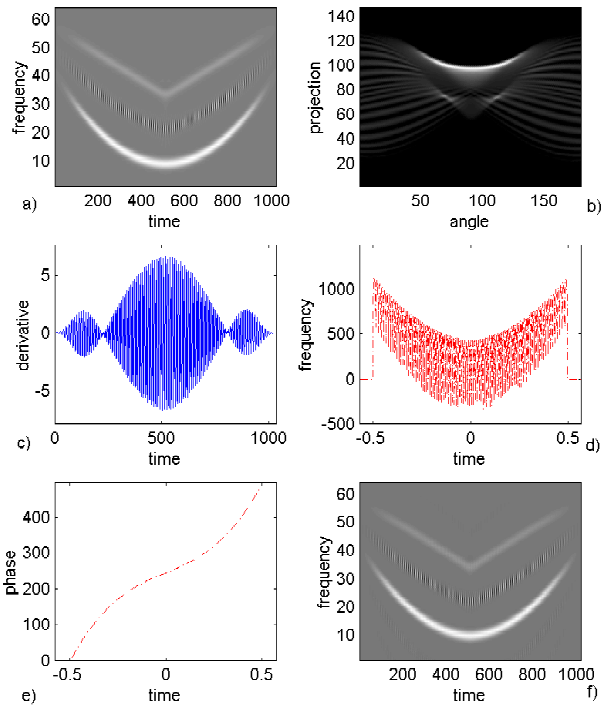

Figure 2: Multicomponent signal and its WD reconstruction from two close fractional power spectra (two WD projections): a) Original WD, b) Projections of the WD (Radon-Wigner transform), c) Derivative approximation: difference of two close projections calculated at 1° and -1°, and divided by the angle step, d) Reconstructed instantaneous frequency of the signal, e) Reconstructed phase of the signal, f) Reconstructed WD.

of the signal's instantaneous frequency, its phase, and the signal itself [Eq. (10)], from these two projections only.

The original WD is given in Fig. 1a. Its Radon-Wigner transform $|X_\alpha(nT)|^2$ [cf. Eq. (3)] is presented in Fig. 1b, for angles $\alpha \in [0^\circ, 180^\circ)$. Only two projections, for $\alpha = \pm 1^\circ$, are used for further calculations. The difference of these projections, $(|X_\alpha(nT)|^2 - |X_{-\alpha}(nT)|^2)/2\alpha$ for $\alpha = 1^\circ$, is shown in Fig. 1c. The reconstructed instantaneous frequency and the reconstructed phase are given in Fig. 1d and Fig. 1e, respectively, by a dash-dot line, while the original, exact values are represented by solid lines. We can see that the agreement between the reconstructed and the original instantaneous frequency is very high. The phase has a constant shift, as can be expected. The reconstructed WD according to (11) is given in Fig. 1f.

Example 2: The reconstruction of a **multicomponent signal**, having the same amplitude variation as the signal in Example 1, but with a different phase variation,

$$x(t) = \exp[-(2.25t)^8]$$
$$\times\{\exp[j\int_{-\infty}^{t}\omega_1(t)dt] + 0.5\exp[j\int_{-\infty}^{t}\omega_2(t)dt]\}, \qquad (13)$$

$$\omega_1(t) = 384\pi\,|t| + 256\pi,\ \omega_2(t) = 1024\pi t^2 + 64\pi,$$

is considered in this example. Note that the instantaneous frequency of this signal is not a continuous function. Nevertheless, we still obtain a satisfactory reconstruction of the phase and the WD, using only two fractional power spectra (see Fig. 2).

Example 3: The reconstruction algorithm is tested for **noisy signals**, as well. The signal from Example 1, contaminated by Gaussian, complex-valued, white noise $\nu(t)$

$$x(t) = \exp[-(2.25t)^8]$$
$$\times\{A\exp[j\int_{-\infty}^{t}(40\pi\sinh^{-1}(100t) + 256\pi t)dt] + \nu(t)\} \quad (14)$$

is considered. Various values of the local signal-to-noise ratio $SNR = 20\log(A/\sigma_\nu)$ have been used in simulations. Figure 3 presents the reconstruction result for a SNR of 9 dB. Small deviations of the reconstructed distribution can be seen in this case. From numerous calculations, we have concluded that the reconstruction threshold is at about $SNR = 3$ dB. Below this value, the degradation of the reconstructed distribution is significant. Nevertheless, it

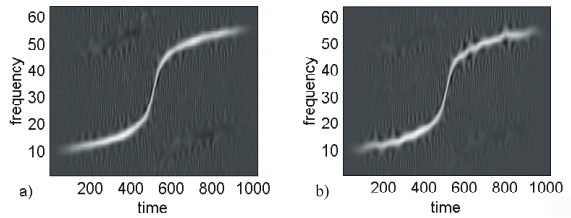

Figure 3: Noisy signal and its WD reconstruction from two close fractional power spectra (two WD projections): a) Original WD, b) Reconstructed WD. Signal to noise ratio is 9 dB.

seems that for signal reconstruction in very high noise, the knowledge of several pairs of close projections would improve the results. In that case we can calculate the differences of the fractional power spectra for different small angles and average them. Furthermore, using other discrete differentiators, different from the simple one given by a mere difference, would also improve noisy case results. However, since the original algorithm produces satisfactory reconstruction even for as low a SNR as a few dB, we have not implemented this variation of the algorithm, for now.

5. CONCLUSIONS

In this paper we have established the connection between the angular derivative of the fractional power spectra and the instantaneous frequency, and we have proposed a method of phase reconstruction from only two close fractional projections of the WD. The numerical simulations show that the discussed phase retrieval algorithm produces good results for different types of signals. The reconstruction technique works well for a signal-to-noise ratio as low as about 3 dB. The main advantages of the proposed method are that it is noniterative and that it demands a minimum number of initial data – only two fractional FT power spectra – which are related to easily measurable intensity distributions. Thus in optics and quantum mechanics, the fractional FT spectrum corresponds to the intensity distribution and probability distribution, respectively.

We have briefly discussed the possible applications of the angular derivatives of the fractional FT power spectra for signal processing, which becomes especially attractive if only the fractional projections of a signal are known.

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