Synthesis of an arbitrary ABCD-system, based on the modified Iwasawa decomposition

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Lossless first-order optical systems

A lossless first-order optical system (or $\mathbf{ABCD}$ system) is described by its symplectic ray transformation matrix. After normalizing this $\mathbf{ABCD}$ matrix,

$$\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}=\begin{bmatrix}\mathbf{w} & \mathbf{0}\\ \mathbf{0} & \mathbf{w}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{c} & \mathbf{d}\end{bmatrix}\begin{bmatrix}\mathbf{w}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{w}\end{bmatrix}, \quad \mathbf{w}=\begin{bmatrix}w_x & 0\\ 0 & w_y\end{bmatrix}$$

with $w_x, w_y > 0$, the normalized $\mathbf{abcd}$ matrix can be decomposed into the so-called modified Iwasawa decomposition

$$\begin{bmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{c} & \mathbf{d}\end{bmatrix}=\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{g} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{s} & \mathbf{0}\\ \mathbf{0} & \mathbf{s}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{x} & \mathbf{y}\\ -\mathbf{y} & \mathbf{x}\end{bmatrix}\equiv\mathcal{L}(\mathbf{g})\,\mathcal{M}(\mathbf{s})\,\mathcal{O}(\mathbf{x}+i\mathbf{y}),$$

where the anamorphic lens $\mathcal{L}(\mathbf{g})$ (e.g., two crossed cylindrical lenses, oriented at the appropriate angle) is described by the symmetric matrix $\mathbf{g} = -(\mathbf{c}\mathbf{a}^t + \mathbf{d}\mathbf{b}^t)(\mathbf{a}\mathbf{a}^t + \mathbf{b}\mathbf{b}^t)^{-1}$,
the separable magnifier $\mathcal{M}(\mathbf{s})$ (using, e.g., two crossed cylindrical lenses with proper orientation again) by the positive-definite symmetric matrix $\mathbf{s} = (\mathbf{a}\mathbf{a}^t + \mathbf{b}\mathbf{b}^t)^{1/2}$,
and the orthosymplectic system $\mathcal{O}(\mathbf{u})$ by the unitary matrix $\mathbf{u} = \mathbf{x} + i\mathbf{y} = (\mathbf{a}\mathbf{a}^t + \mathbf{b}\mathbf{b}^t)^{-1/2}(\mathbf{a} + i\mathbf{b})$. $\mathcal{O}(\mathbf{u})$ can be realized in many ways, for instance as a separable fractional Fourier transformer (with different fractional angles in the two perpendicular directions) embedded in between two rotators (with different rotation angles): $\mathcal{R}(\beta)\,\mathcal{F}(\gamma_x, \gamma_y)\,\mathcal{R}(\alpha)$.

One-dimensional case

The basic 2×2 ray transformation matrices

$$\begin{bmatrix}1 & \lambda z\\ 0 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 0\\ -1/\lambda f & 1\end{bmatrix}, \quad \begin{bmatrix}\cos\gamma & w^2\sin\gamma\\ -w^{-2}\sin\gamma & \cos\gamma\end{bmatrix}$$

correspond to a section of free space $\mathcal{S}(z)$ with distance z, a cylindrical lens $\mathcal{L}(f)$ with focal length f, and a fractional Fourier transformer $\mathcal{F}(\gamma; w)$ with fractional angle γ and scaling w, respectively, acting on light with wavelength λ.
The Iwasawa decomposition leads to the actual realization

$$\begin{bmatrix}A & B\\ C & D\end{bmatrix}=\begin{bmatrix}1 & 0\\ -gw^{-2} & 1\end{bmatrix}\begin{bmatrix}s & 0\\ 0 & s^{-1}\end{bmatrix}\begin{bmatrix}\cos\gamma & w^2\sin\gamma\\ -w^{-2}\sin\gamma & \cos\gamma\end{bmatrix}$$

and in operator notation $\boxed{\mathcal{L}(w^2/\lambda g)\,\mathcal{M}(s)\,\mathcal{F}(\gamma; w)}$.
A magnifier $\mathcal{M}(s)$ (with reversion) can be realized as

$$\boxed{\mathcal{L}(z_\circ - f_\circ)\,\mathcal{S}(z_\circ)\,\mathcal{L}(f_\circ)\,\mathcal{S}(d_\circ)}$$

with $1/d_\circ + 1/z_\circ = 1/f_\circ$ and $s = z_\circ/d_\circ$.
For the fractional Fourier transformer $\mathcal{F}(\gamma; w)$ we may use one of the two Lohmann setups for which $\sin^2(\gamma/2) = \tilde{d}/2\tilde{f}$:

(a) $\mathcal{F}_a(\gamma; \tilde{w}) = \mathcal{S}(\tilde{d})\,\mathcal{L}(\tilde{f})\,\mathcal{S}(\tilde{d})$ with $\tilde{w}^2\tan(\gamma/2) = \lambda\tilde{d}$
(b) $\mathcal{F}_b(\gamma; \tilde{w}) = \mathcal{L}(\tilde{f})\,\mathcal{S}(\tilde{d})\,\mathcal{L}(\tilde{f})$ with $\tilde{w}^2\sin\gamma = \lambda\tilde{d}$

$$\boxed{\mathcal{M}(w\tilde{w}^{-1})\,\mathcal{F}(\gamma; \tilde{w})\,\mathcal{M}(\tilde{w}w^{-1})}$$

Note the scale adapter $\mathcal{M}(w\tilde{w}^{-1})$ and the freedom for $\tilde{w}$.
The final one-dimensional cascade, using setup (a), leads to

$$\boxed{\mathcal{L}(w^2/\lambda g)\,\mathcal{L}(z_\circ - f_\circ)\,\mathcal{S}(z_\circ)\,\mathcal{L}(f_\circ)\,\mathcal{S}(d_\circ)\,\mathcal{S}(\tilde{d})\,\mathcal{L}(\tilde{f})\,\mathcal{S}(\tilde{d})}$$

with three lenses and three sections of free space; note that the final lens $\mathcal{L}(f)$ with $1/f = \lambda g/w^2 + 1/(z_\circ - f_\circ)$ takes also care of the phase correction that is required for the magnifier.

Two-dimensional case

In the two-dimensional case, two new elements will enter the description: a rotator $\mathcal{R}(\varphi)$ and a scaling magnifier $\mathcal{M}_\circ = \mathcal{M}(w_x^{1/2}w_y^{-1/2}, w_x^{-1/2}w_y^{1/2})$ (necessary if $w_x \neq w_y$), with

$$\mathbf{u}_r(\varphi)=\begin{bmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{bmatrix} \quad \mathbf{s}_\circ=\begin{bmatrix}w_x^{1/2}w_y^{-1/2} & 0\\ 0 & w_x^{-1/2}w_y^{1/2}\end{bmatrix}.$$

The matrix $\mathbf{G}$ of the anamorphic lens can be written as

$$\mathbf{G} = \mathbf{w}^{-1}\mathbf{g}\mathbf{w}^{-1} = \begin{bmatrix}g_{11} & g_{12}\\ g_{12} & g_{22}\end{bmatrix} = \mathbf{u}_r(\varphi_g)\begin{bmatrix}g_1 & 0\\ 0 & g_2\end{bmatrix}\mathbf{u}_r(-\varphi_g)$$

and leads to a combination of two crossed cylindrical lenses whose focal lengths read $1/\lambda g_1$ and $1/\lambda g_2$, and which is oriented at an angle φ_g: $\boxed{\mathcal{R}(\varphi_g)\mathcal{L}(1/\lambda g_1, 1/\lambda g_2)\mathcal{R}(-\varphi_g)}$.
For the separable magnifier we have $\mathbf{w}\mathbf{s}\mathbf{w}^{-1} = \mathbf{s}_\circ\mathbf{s}\mathbf{s}_\circ^{-1}$, which leads to $\boxed{\mathcal{M}_\circ\,\mathcal{R}(\varphi_s)\,\mathcal{M}(s_1, s_2)\,\mathcal{R}(-\varphi_s)\,\mathcal{M}_\circ^{-1}}$.
Two crossed 1D magnifiers, with different magnifications $s_1 = z_1/d_1$ and $s_2 = z_2/d_2$ in the two perpendicular directions, and with $z_1 + d_1 = z_2 + d_2$, lead to a separable magnifier.
We also have $\mathbf{w}\mathbf{u}_r(\varphi)\mathbf{w}^{-1} = \mathbf{s}_\circ\mathbf{u}_r(\varphi)\mathbf{s}_\circ^{-1}$, so that a rotator embedded in between scaling magnifiers leads to

$$\boxed{\mathcal{M}_\circ\,\mathcal{R}(\varphi)\,\mathcal{M}_\circ^{-1}}.$$

Finally, the separable fractional Fourier transformer

$$\begin{bmatrix}\cos\gamma_x & 0 & w_x^2\sin\gamma_x & 0\\ 0 & \cos\gamma_y & 0 & w_y^2\sin\gamma_y\\ -w_x^{-2}\sin\gamma_x & 0 & \cos\gamma_x & 0\\ 0 & -w_y^{-2}\sin\gamma_y & 0 & \cos\gamma_y\end{bmatrix}$$

is simply a concatenation of two 1D fractional Fourier transformers. And like before, we have

$$\boxed{\mathcal{M}(w_x\tilde{w}_x^{-1}, w_y\tilde{w}_y^{-1})\,\mathcal{F}(\gamma_x, \gamma_y; \tilde{w}_x, \tilde{w}_y)\,\mathcal{M}(\tilde{w}_x w_x^{-1}, \tilde{w}_y w_y^{-1})}.$$

From a practical point of view we may require that the input and output planes of the separable fractional Fourier transformer for the x and the y coordinates coincide: $\tilde{d}_x = \tilde{d}_y = \tilde{d}$. This leads to $\tilde{w}_x^2\tan(\gamma_x/2) = \tilde{w}_y^2\tan(\gamma_y/2)$ for setup (a) and to $\tilde{w}_x^2\sin\gamma_x = \tilde{w}_y^2\sin\gamma_y$ for setup (b), where we have tacitly assumed that $\gamma_{x,y} \neq 0$. We thus need freedom for $\tilde{w}_{x,y}$.
After choosing $w_x = w_y = w$, the final cascade takes the form

$$\mathcal{R}(\varphi_g)\,\mathcal{L}(1/\lambda g_1, 1/\lambda g_2)\,\mathcal{R}(\varphi_s - \varphi_g)\,\mathcal{M}(s_1, s_2)\,\mathcal{R}(\beta - \varphi_s)$$
$$\mathcal{M}(w\tilde{w}_x^{-1}, w\tilde{w}_y^{-1})\,\mathcal{F}(\gamma_x, \gamma_y; \tilde{w}_x, \tilde{w}_y)\,\mathcal{M}(\tilde{w}_x w^{-1}, \tilde{w}_y w^{-1})\,\mathcal{R}(\alpha).$$

The number of cylindrical lenses that we need, equals ten: two lenses for the anamorphic lens (which at the same time take care of the phase corrections for the separable magnifier); two lenses to realize the separable magnifier; two lenses for the separable fractional Fourier transformer; and two lenses for each of the scale adapters that embed the fractional Fourier transformer.