On the moments of the Wigner distribution and the fractional Fourier transform

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Signal Processing Systems

Wigner distribution

The Wigner distribution of a signal $x(t)$ is defined as

$$W_x(t,f) = \int_{-\infty}^{\infty} x(t+\tfrac{1}{2}\tau)x^*(t-\tfrac{1}{2}\tau)\exp(-j2\pi f\tau)d\tau.$$

The real-valued WD can roughly be considered as the signal's energy distribution over the time-frequency plane.

Fractional Fourier transform

The fractional Fourier transform is defined as

$$X_\alpha(u) = \mathcal{R}^\alpha\left[x(t)\right](u) = \int_{-\infty}^{\infty} K(\alpha,t,u)x(t)dt,$$

where the kernel $K(\alpha,t,u)$ is given by

$$K(\alpha,t,u) = \frac{\exp(j\frac{1}{2}\alpha)}{\sqrt{j\sin\alpha}}\exp\left(j\pi\frac{(t^2+u^2)\cos\alpha - 2ut}{\sin\alpha}\right)$$

The fractional FT can be considered as a generalization of the ordinary FT; for $\alpha = \frac{1}{2}\pi$ and $\alpha = -\frac{1}{2}\pi$ it reduces to the ordinary and inverse FT, respectively.
A fractional Fourier transformation produces a rotation of quadratic time-frequency representations like the WD in the time-frequency plane:

$$\begin{array}{lcl} x(t) & \longleftrightarrow & W_x(t,f) \\ \downarrow \text{fractional FT} & & \downarrow \text{rotation of WD} \\ X_\alpha(t) = \mathcal{R}^\alpha\left[x\right] & \longleftrightarrow & W_{X_\alpha}(t,f), \end{array}$$

with $W_{X_\alpha}(t,f) = W_x(t\cos\alpha - f\sin\alpha, t\sin\alpha + f\cos\alpha)$.
Hence, for the fractional power spectrum, we have

$$|X_\alpha(t)|^2 = \int_{-\infty}^{\infty} W_x(t\cos\alpha - f\sin\alpha, t\sin\alpha + f\cos\alpha)df.$$

Fractional Fourier transform moments

By analogy with the well-known time and frequency moments, the fractional FT moments can be introduced:

$$\int_{-\infty}^{\infty} t^n\left|X_\alpha(t)\right|^2 dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t^n W_{X_\alpha}(t,f)dtdf.$$

The zero-order moment E represents the signal's energy and is invariant under fractional Fourier transformation; this is Parseval's theorem for unitary transformations.
The normalized first-order moment m_α is related to the center of gravity of the fractional power spectrum. From the simple relationship $m_\alpha = m_0\cos\alpha + m_{\pi/2}\sin\alpha$, it is easy to see that the sum of squares $m_\alpha^2 + m_{\alpha+\pi/2}^2$ is invariant under fractional Fourier transformation.
The normalized second-order moment w_α is related to the effective width of the signal in the fractional FT domain, while the normalized **mixed** second-order fractional FT moment μ_α – with $tfW_{X_\alpha}(t,f)$ instead of $t^2W_{X_\alpha}(t,f)$ in the integrand – takes the form

$$\mu_\alpha = \frac{1}{4\pi jE}\int_{-\infty}^{\infty}\left[X_\alpha^*(t)\frac{\partial X_\alpha(t)}{\partial t} - X_\alpha(t)\frac{\partial X_\alpha^*(t)}{\partial t}\right]tdt.$$

From the relations between the second-order moments,

$$\begin{aligned} w_\alpha &= w_0\cos^2\alpha + w_{\pi/2}\sin^2\alpha - \mu_0\sin 2\alpha \\ \mu_\alpha &= \tfrac{1}{2}(w_0 - w_{\pi/2})\sin 2\alpha + \mu_0\cos 2\alpha, \end{aligned}$$

we conclude (i) that all second-order moments w_α and μ_α can be obtained from any three second-order moments w_α, (ii) that $w_\alpha + w_{\alpha+\pi/2}$ is invariant under fractional Fourier transformation, and (iii) that $\mu_\alpha = 0$ corresponds to the fractional domain with the extremum signal width w_α. We also have that $w_\alpha w_{\alpha+\pi/2} = w_0 w_{\pi/2} + \frac{1}{4}[(w_0 - w_{\pi/2})^2 - 4\mu_0^2]\sin^2 2\alpha + \frac{1}{2}\mu_0(w_0 - w_{\pi/2})\sin 4\alpha$, which expression is, in general, not invariant under fractional Fourier transformation, and which has a lower bound of $\frac{1}{4}$, due to the uncertainty principle.
Instead of **global** moments, which we considered above, one can consider **local** moments, which are related to such things as the instantaneous power and instantaneous frequency (for $\alpha = 0$) or the spectral energy density and group delay (for $\alpha = \frac{1}{2}\pi$) in the different fractional FT domains. The local frequency $U_{X_\alpha}(t)$ in the fractional FT domain with parameter α is defined as

$$U_{X_\alpha}(t) = \int_{-\infty}^{\infty} fW_{X_\alpha}(t,f)df \bigg/ \int_{-\infty}^{\infty} W_{X_\alpha}(t,f)df$$

$$= \frac{1}{2\left|X_\alpha(t)\right|^2}\int_{-\infty}^{\infty}\left.\frac{\partial|X_\beta(\tau)|^2}{\partial\beta}\right|_{\beta=\alpha}\text{sgn}(\tau - t)d\tau$$

and is related to the phase $\varphi_\alpha(t) = \arg X_\alpha(t)$ of the fractional FT $X_\alpha(t)$ through $U_{X_\alpha}(t) = (1/2\pi)d\varphi_\alpha(t)/dt$. This implies that the derivative of the fractional power spectra with respect to the angle α defines the local frequency in the fractional domain, and that it can be used for solving the phase retrieval problem by measuring intensity functions only.