Phase Reconstruction from Two Wigner Distribution Projections

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Fractional power spectra

The fractional Fourier transform of $x(t)$ is defined as

$$X_\alpha(u) = \int_{-\infty}^{\infty} K(\alpha, t, u) x(t) dt, \qquad (1)$$

where the kernel $K(\alpha, t, u)$ is given by

$$K(\alpha, t, u) = \frac{\exp(j\frac{1}{2}\alpha)}{\sqrt{j \sin\alpha}} \exp\left[j\pi \frac{(t^2 + u^2)\cos\alpha - 2ut}{\sin\alpha}\right].$$

The fractional power spectra $|X_\alpha(u)|^2$ are equal to the projections of the Wigner distribution (WD) $W_x(t, f)$,

$$|X_\alpha(u)|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_x(t, f)\delta(t\cos\alpha + f\sin\alpha - u) df\, dt$$

$$= \int_{-\infty}^{\infty} W_x(u\cos\alpha - f\sin\alpha, u\sin\alpha + f\cos\alpha) df, \qquad (2)$$

and equal to the Fourier transform of the ambiguity function:

$$|X_\alpha(u)|^2 = \int_{-\infty}^{\infty} A_x(R\sin\alpha, -R\cos\alpha) e^{-j2\pi R u} du. \qquad (3)$$

Projection and instantaneous frequency

For the instantaneous frequency $f_0(t)$ at time t we have

$$f_0(t) = \frac{1}{2\pi j}\frac{1}{|x(t)|^2}\int_{-\infty}^{\infty} \left.\frac{\partial A_x(\tau, \nu)}{\partial \tau}\right|_{\tau=0} e^{j2\pi t\nu} d\nu$$

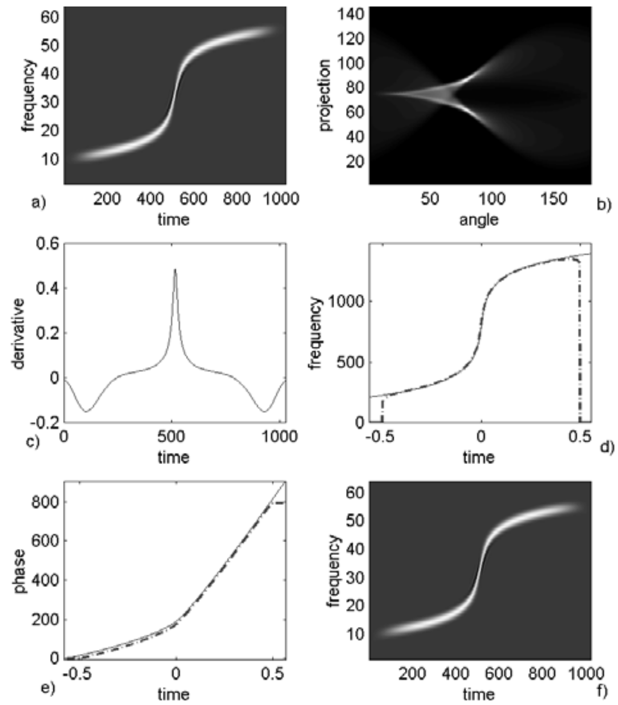
$$= \frac{-1}{2\,|X_0(t)|^2}\int_{-\infty}^{\infty} \left.\frac{\partial\,|X_\alpha(u)|^2}{\partial\alpha}\right|_{\alpha=0} \mathrm{sgn}(t-u) du. \qquad (4)$$

We thus get for the signal $x(t) = |X_0(t)|\exp[j\varphi(t)]$, that its phase derivative $\varphi'(t) = d\varphi(t)/dt = 2\pi f_0(t)$ is determined by its intensity $|X_0(t)|^2$ and the convolution of the signum function with the angular derivative of the fractional power spectrum $\partial|X_\alpha(u)|^2/\partial\alpha$ at the angle $\alpha = 0$. This implies that only two fractional power spectra for close angles suffice to solve the signal retrieval problem, up to a constant phase factor.

Computer simulation

After two WD projections have been obtained, the instantaneous frequency is calculated as the output of the linear system, cf. Eq. (4),

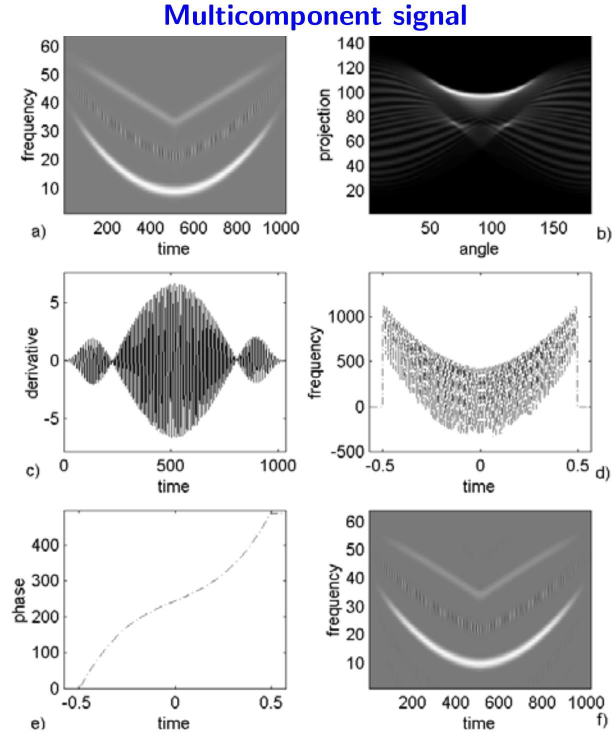
$$f_0(nT) = -\frac{\dfrac{|X_\alpha(nT)|^2 - |X_{-\alpha}(nT)|^2}{2\alpha} *_n \mathrm{sgn}(nT)}{2\,|X_0(nT)|^2}\, T.$$



Monocomponent signal

a) Original WD, b) Projections of the WD, c) Derivative approximation: difference of two close projections calculated at 1° and -1°, and divided by the angle step, d) Reconstructed (dash-dot) and original (solid line) instantaneous frequency of the signal, e) Reconstructed (dash-dot) and original (solid line) phase of the signal, f) Reconstructed WD.

Multicomponent signal



The reconstruction technique works well for a signal-to-noise ratio as low as about 3 dB.