Wigner Distribution Weighted in the Fractional Fourier Domain

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Weighted Wigner Distribution

The pseudo Wigner distribution $PWD_f(t,\omega)$ of a (multi-component) signal $f(x)$ can be defined as

$$PWD_f(t,\omega) = \int_{-\infty}^{\infty} ST_f^0(t,\omega+\theta) ST_f^{0*}(t,\omega-\theta)d\theta, \quad (1)$$

where

$$ST_f^0(t,\omega) = \int_{-\infty}^{\infty} f(t+x)g^*(x)e^{-j2\pi x\omega}dx \quad (2)$$

is the short-time Fourier transform (STFT).

The SWWD, the weighted form of (1), has been introduced for multicomponent signals, to produce a representation close to the sum of the WDs of each component separately, but with reduced cross-terms:

$$P_f^0(t,\omega) = \int_{-\infty}^{\infty} ST_f^0(t,\omega+\theta)z(\theta)ST_f^{0*}(t,\omega-\theta)d\theta. \quad (3)$$

Fractional Fourier transform domain

We consider a rotation of the (t,ω) coordinate system, which corresponds to a fractional FT of $f(x)$,

$$R_f^\alpha(u) = \int_{-\infty}^{\infty} K(\alpha,x,u)f(x)dx, \quad (4)$$

where the kernel $K(\alpha,x,u)$ is given by

$$K(\alpha,x,u) = \frac{\exp(j\frac{1}{2}\alpha)}{\sqrt{j\sin\alpha}} \exp\left[j\pi\frac{(x^2+u^2)\cos\alpha - 2ux}{\sin\alpha}\right].$$

The fractional STFT $ST_f^\alpha(t,\omega)$ is defined as the STFT of the fractional FT $R_f^\alpha(x)$ of the signal:

$$ST_f^\alpha(t,\omega) = \int_{-\infty}^{\infty} R_f^\alpha(t+x)g^*(x)e^{-j2\pi x\omega}dx. \quad (5)$$

From the symmetry property $R_{g^*}^\alpha(u) = [R_g^{-\alpha}(u)]^*$ of the fractional FT, and from the relations between the fractional FT and the STFT, we have the relationship

$$e^{j\pi t\omega}\int_{-\infty}^{\infty} R_f^\alpha(x)g^*(t-x)e^{-j2\pi x\omega}dx \quad (6)$$

$$= e^{j\pi uv}\int_{-\infty}^{\infty} f(x)[R_g^{-\alpha}(u-x)]^* e^{-j2\pi xv}dx, \quad (7)$$

with $u = t\cos\alpha - \omega\sin\alpha, v = t\sin\alpha + \omega\cos\alpha$. (8)

The SWWD in the fractional domain, is defined as

$$P_f^\alpha(t,\omega) = \int_{-\infty}^{\infty} ST_f^\alpha(u,v+\theta)z(\theta)ST_f^{\alpha*}(u,v-\theta)d\theta. \quad (9)$$

Signal width and fractional moments

The signal width in any fractional α-domain is related to its second-order central fractional FT moment:

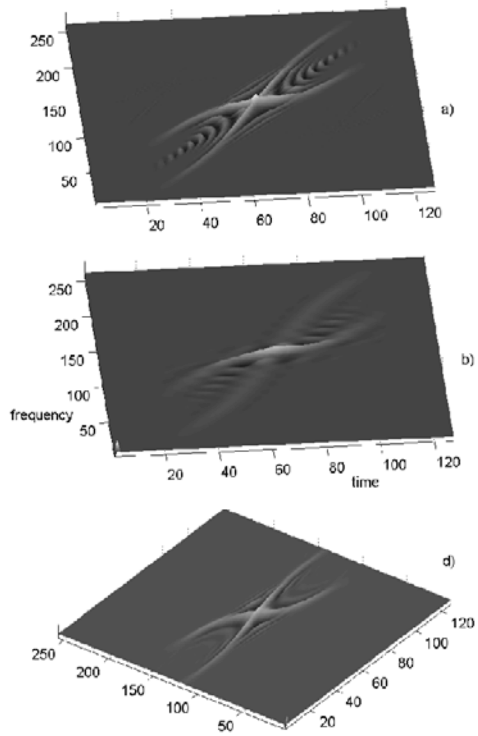
$$p_\alpha = \frac{1}{E}\int_{-\infty}^{\infty} |R_f^\alpha(x)|^2(x-m_\alpha)^2dx = \frac{(w_\alpha - m_\alpha^2)}{E}. \quad (10)$$

Any second-order moment w_α can be obtained from three others; we have, for instance, the relationship

$$\begin{aligned} p_\alpha E &= p_0\cos^2\alpha + p_{\pi/2}\sin^2\alpha \quad (11)\\ &\quad + [w_{\pi/4} - m_0 m_{\pi/2} - \tfrac{1}{2}(w_0 + w_{\pi/2})]\sin 2\alpha. \end{aligned}$$

Thus, the fractional domain where the signal is the best concentrated – i.e., first derivative of p_α equals zero – can be found from the knowledge of only three fractional power spectra, at 0, $\frac{1}{2}\pi$, and $\frac{1}{4}\pi$, for instance. We can expect that the application of the SWWD in this domain will be the most efficient one.

Computer simulation



The standard WD is shown in (a). The SWWD calculated by the standard definition, i.e., along the frequency axis is presented in (b). The SWWD based on the fractional FT for the optimal α is presented in (d). We can see that, as a consequence of the high concentration of the components along the optimal fractional angle, we almost achieved the goal of getting the auto-terms of the WD without any cross-terms.