# Evolution of the vortex part of the orbital angular momentum in separable first-order optical systems

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## Wigner distribution

The Wigner distribution $W(x,u;y,v)$ represents partially coherent light in a combined space/spatial-frequency domain, the so-called phase space.

## Wigner distribution moments

With $E = \iiiint W(x,u;y,v)\,dx\,du\,dy\,dv$ the total energy of the light, we define the normalized moments $\mu_{pqrs}$ as

$$\mu_{pqrs}\,E = \iiiint_{-\infty}^{\infty} W(x,u;y,v)\,x^p u^q y^r v^s\,dx\,du\,dy\,dv.$$

The ten second-order moments are usually combined into

$$\boldsymbol{M} = \begin{bmatrix} \mu_{2000} & \mu_{1010} & \mu_{1100} & \mu_{1001} \\ \mu_{1010} & \mu_{0020} & \mu_{0110} & \mu_{0011} \\ \mu_{1100} & \mu_{0110} & \mu_{0200} & \mu_{0101} \\ \mu_{1001} & \mu_{0011} & \mu_{0101} & \mu_{0002} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{P} \\ \boldsymbol{P}^t & \boldsymbol{Q} \end{bmatrix}.$$

## Orbital angular momentum

The OAM $\Lambda$ of an optical beam can be expressed as

$$\Lambda = \mu_{1001} - \mu_{0110}.$$

The asymmetrical part $\Lambda_a$ and the vortex part $\Lambda_v$ read

$$\begin{aligned} \Lambda_a &= [(\mu_{2000}-\mu_{0020})(\mu_{1001}+\mu_{0110}) \\ &\quad - 2\mu_{1010}(\mu_{1100}-\mu_{0011})]/(\mu_{2000}+\mu_{0020}) \\ \Lambda_v &= 2T/(\mu_{2000}+\mu_{0020}), \end{aligned}$$

respectively, with $T$ – the optical twist – defined as

$$T = \mu_{0020}\mu_{1001} - \mu_{2000}\mu_{0110} + \mu_{1010}(\mu_{1100}-\mu_{0011}).$$

## Moments evolution in separable systems

The input-output relationship of a separable system reads

$$\begin{aligned} W_{\text{out}}(x,u;y,v) = W_{(}d_x x - b_x u, -c_x x + a_x u; \\ d_y y - b_y v, -c_y y + a_y v), \end{aligned}$$

with $a_x d_x - b_x c_x = 1$ and $a_y d_y - b_y c_y = 1$.
The input-output relationship for the moments reads

$$\begin{aligned} \mu_{pqrs}^{\text{out}} &= \sum_{k=0}^{p}\sum_{l=0}^{q}\sum_{m=0}^{r}\sum_{n=0}^{s} \binom{p}{k}\binom{q}{l}\binom{r}{m}\binom{s}{n} a_x^{p-k} \\ &\times b_x^k c_x^l d_x^{q-l} a_y^{r-m} b_y^m c_y^n d_y^{s-n} \mu_{p-k+l,q-l+k,r-m+n,s-n+m}. \end{aligned}$$

For Fourier transforming systems we have $T^{\text{out}}|_{FT_x} = b_x\,T_x$, $T^{\text{out}}|_{FT_y} = b_y\,T_y$, and $T^{\text{out}}|_{FT_{xy}} = b_x b_y\,T_{xy}$, with

$$\begin{aligned} T_x &= \mu_{0020}\mu_{0101} + \mu_{0200}\mu_{1010} - \mu_{0110}(\mu_{1100}+\mu_{0011}), \\ T_y &= \mu_{0002}\mu_{1010} - \mu_{2000}\mu_{0101} + \mu_{1001}(\mu_{1100}+\mu_{0011}), \\ T_{xy} &= \mu_{0002}\mu_{0110} + \mu_{0200}\mu_{1001} - \mu_{0101}(\mu_{1100}-\mu_{0011}). \end{aligned}$$

## Evolution of twist and vortex part of OAM

The OAM $\Lambda^{\text{out}}$ at the output of a separable system reads

$$\begin{aligned} \Lambda^{\text{out}} &= \mu_{1001}(a_x d_y - b_y c_x) - \mu_{0110}(a_y d_x - b_x c_y) \\ &\quad + \mu_{0101}(b_x d_y - b_y d_x) + \mu_{1010}(a_x c_y - a_y c_x), \end{aligned}$$

while the output twist takes the form

$$T^{\text{out}} = a_x a_y T + a_y b_x T_x + a_x b_y T_y + b_x b_y T_{xy}.$$

We further need the expression

$$\begin{aligned} \mu_{2000}^{\text{out}} + \mu_{0020}^{\text{out}} &= a_x^2\mu_{2000} + 2a_x b_x \mu_{1100} + b_x^2\mu_{0200} \\ &\quad + a_y^2\mu_{0020} + 2a_y b_y \mu_{0011} + b_y^2 \mu_{0002}. \end{aligned}$$

We conclude that the evolution of the vortex part $\Lambda_v$ depends only on $a_x$, $a_y$, $b_x$, and $b_y$.

## Special case: isotropic systems

For an isotropic system we have $\Lambda^{\text{out}} = \Lambda$; this does not hold for $\Lambda_a$ and $\Lambda_v$. We have (with $p = b/a$)

$$\begin{aligned} \Lambda_v^{\text{out}} = \Lambda_v(p) &= 2\,[T + p(T_x+T_y) + p^2 T_{xy}] \\ &\times\ [(\mu_{2000}+\mu_{0020}) + 2p(\mu_{1100}+\mu_{0011}) \\ &\quad + p^2(\mu_{0200}+\mu_{0002})]^{-1}. \end{aligned}$$

- For the special signal satisfying

$$\frac{T}{\mu_{2000}+\mu_{0020}} = \frac{T_x+T_y}{2(\mu_{1100}+\mu_{0011})} = \frac{T_{xy}}{\mu_{0200}+\mu_{0002}},$$

we have $\Lambda_v^{\text{out}} = \Lambda_v$. This holds in particular for rotationally symmetric beams: $\Lambda_a$ remains zero.

- Between conjugate planes ($b=0$): $\Lambda_v^{\text{out}} = \Lambda_v$.
- Optical systems with the same $p$ behave similarly with respect to the evolution of $\Lambda_v$.
- One can find the values of the parameter $p$ where $\Lambda_v(p)$ is zero, or has maxima or minima. Note that, while the denominator is positive for all possible values of $p$, the numerator – and therefore $\Lambda_v(p)$ itself – may or may not change its sign (but not more than twice) depending upon the actual values of $T$, $T_x+T_y$, and $T_{xy}$.
- $p = \{-(T_x+T_y) \pm [(T_x+T_y)^2 - 4TT_{xy}]^{1/2}\}/2T_{xy}$ yields $\Lambda_v(p) = 0$, and since $p$ has to be real, this is possible only if the condition $(T_x+T_y)^2 \geq 4TT_{xy}$ holds.